SOME SMOOTH FINSLER DEFORMATIONS OF HYPERBOLIC SURFACES

BRUNO COLBOIS, FLORENCE NEWBERGER, AND PATRICK VEROVIC

Abstract. Given a closed hyperbolic Riemannian surface, the aim of the present paper is to describe an explicit construction of *smooth* deformations of the hyperbolic metric into Finsler metrics that are *not Riemannian* and whose properties are such that the classical Riemannian results about entropy rigidity, marked length spectrum rigidity and boundary rigidity all *fail* to extend to the Finsler category.

1. Introduction

In this paper, we construct Finsler metrics on hyperbolic surfaces, proving that certain recent Riemannian rigidity results fail to extend to the Finsler category. Recall that a Finsler metric $\mathcal{F}$ on a manifold M is a continuous function $\mathcal{F} : TM \longrightarrow \mathbf{R}$ such that $\mathcal{F}(p, \cdot)$ is a norm on T_pM for any $p \in M$. If in addition $\mathcal{F}$ is C^∞ on $TM \smallsetminus \{0\}$ (the tangent bundle minus the zero section), then $\mathcal{F}$ is said to be *smooth*. In that case, $\mathcal{F}$ is called *strongly convex* iff for any $(p, v) \in TM \smallsetminus \{0\}$, the symmetric bilinear form $\frac{\partial^2 \mathcal{F}^2}{\partial v^2}(p, v)$ on T_pM is positive definite.

Note that a Riemannian metric g on M gives rise to an associated Finsler metric $\mathcal{F}$ defined by $\mathcal{F}(p, v) = (g(p){\cdot}(v, v))^{1/2}$. We will then say that $\mathcal{F}$ is Riemannian.

Perhaps the most significant Riemannian rigidity results are the minimal entropy rigidity theorems of Katok for surfaces and of Besson, Courtois, and Gallot in higher dimensions, due to their many applications (see for example the surveys [3] and [8]). We begin by stating these two theorems and one of their consequences, relevant to our work (for a more general version in higher dimensions, see [4]).

Given a Finsler metric F on a simply connected manifold $\tilde{M}$, the volume growth entropy of F, denoted by $h(F)$, is the asymptotic exponential growth rate of F-balls in $\tilde{M}$, *i.e.*,

$$h(F) := \limsup_{R \to +\infty} \frac{1}{R} \log\left(\mathrm{Vol}_F(B_F(x, R))\right) \in [0, +\infty]$$

for an arbitrary $x \in \tilde{M}$, where $B_F(x, R)$ is the open ball of radius R in $\tilde{M}$ about x with respect to F, and Vol_F denotes the Holmes-Thompson volume on $\tilde{M}$ associated with F (see Section 2 for the definition).

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By extending this definition, if $\mathcal{F}$ is a Finsler metric on a *compact* manifold M whose universal cover is $\tilde{M}$, the upper limit above is actually a limit, and the volume growth entropy $h(\mathcal{F})$ of $\mathcal{F}$ is defined to be equal to that $h(F)$ of the lift F to $\tilde{M}$ of the metric $\mathcal{F}$.

Theorem 1.1 (Katok, [16])**.** *Let (S, g_0) be a closed hyperbolic (Riemannian) surface, and let g be a Riemannian metric on S. Then, denoting by* vol *the usual Riemannian volume, we have*

(1) $h(g_0)^2 \operatorname{vol}_{g_0}(S) \leqslant h(g)^2 \operatorname{vol}_g(S)$, and

(2) $h(g_0)^2 \operatorname{vol}_{g_0}(S) = h(g)^2 \operatorname{vol}_g(S)$ if and only if g is hyperbolic.

Later, Besson, Courtois and Gallot extended the first part (inequality) of this result to higher dimensions and obtained something different for the second part (rigidity):

Theorem 1.2 (Besson–Courtois–Gallot, [3] and [4])**.** *Let (M, g_0) be a closed n-dimensional Riemannian locally symmetric space of negative curvature with $n \geqslant 3$, and let (N, g) be a closed negatively curved Riemannian manifold homotopy equivalent to (M, g_0). Then, denoting by* vol *the usual Riemannian volume, we have*

(1) $h(g_0)^n \operatorname{vol}_{g_0}(M) \leqslant h(g)^n \operatorname{vol}_g(N)$, and

(2) $h(g_0)^n \operatorname{vol}_{g_0}(M) = h(g)^n \operatorname{vol}_g(N)$ if and only if (N, g) is homothetic to (M, g_0).

An important corollary of these results is the boundary rigidity of negatively curved symmetric spaces (see [8] for a general discussion of boundary rigidity). Let (M, g_0) denote a compact connected Riemannian manifold with a non-empty boundary ∂M, and let d_{g_0} denote the induced metric on ∂M by the distance function on M associated with g_0. Such a manifold is called boundary rigid if and only if for any compact connected Riemannian manifold (N, g) with non-empty boundary ∂N, any *metric* isometry $(\partial M, d_{g_0}) \longrightarrow (\partial N, d_g)$ extends to a *smooth* isometry $(M, g_0) \longrightarrow (N, g)$, where d_g is defined the same way as d_{g_0}.

Theorem 1.3 ([8], Corollary 6.3)**.** *Any bounded domain in a Riemannian symmetric space of negative curvature has a closure that is boundary rigid.*

Another Riemannian rigidity result is related to the marked length spectrum. The marked length spectrum of a Finsler manifold $(M, \mathcal{F})$ is the map that assigns to each free homotopy class of M the $\mathcal{F}$-length of a shortest closed parameterized curve $[0, 1] \longrightarrow M$ (thus a $\mathcal{F}$-geodesic since it is locally $\mathcal{F}$-length minimizing) in that free homotopy class.

Theorem 1.4 ([9], Theorem 1.1. See also [8], Theorem 8.2)**.** *Let g_0 be a negatively curved Riemannian metric on a closed manifold M. Let $(g_\lambda)_{\lambda \in (-\varepsilon, \varepsilon)}$ be a smooth variation of g_0 through Riemannian metrics on M such that for each $\lambda \in (-\varepsilon, \varepsilon)$, the marked length spectrum of g_λ is the same as that of g_0. Then g_λ is isometric to g_0 for all $\lambda \in (-\varepsilon, \varepsilon)$.*

We now state our main result.

Main Theorem. *Let $\mathcal{D}$ denote an open ball in the two-dimensional hyperbolic space $(\mathbf{H}^2, g_0)$. Then there exist $\varepsilon > 0$ and a continuous function $\Phi : (-\varepsilon, \varepsilon) \times T\mathbf{H}^2 \longrightarrow \mathbf{R}$ that is C^∞ on $(-\varepsilon, \varepsilon) \times (T\mathbf{H}^2 \smallsetminus \{0\})$ and such that for each $\lambda \in (-\varepsilon, \varepsilon)$, we have*

(1) $\mathsf{F}_\lambda(\cdot) := \Phi(\lambda, \cdot)$ *is a* smooth *strongly convex Finsler metric on* $\mathbf{H}^2$,

(2) F_0 *is associated with* g_0, *and* F_λ *is* not Riemannian *whenever* $\lambda \neq 0$,

(3) $d_{\mathsf{F}_\lambda} = d_{\mathsf{F}_0}$, *where* d_{F_0} *and* d_{F_λ} *are the metrics induced on* $\partial\mathcal{D}$ *by the distance functions on* $\mathbf{H}^2$ *associated respectively with* F_0 *and* F_λ,

(4) *every two points in* $\mathcal{D}$ *can be joined by a geodesic of* F_λ *whose image is contained in* $\mathcal{D}$,

(5) $\mathsf{F}_\lambda(x, u) = \mathsf{F}_0(x, u)$ *for all* $x \in \mathbf{H}^2 \smallsetminus \mathcal{D}$ *and* $u \in T_x\mathbf{H}^2$,

(6) F_λ *has no conjugate points.*

It is important to point out that the greatest difficulty in the proof of this theorem is the *smoothness* property we expect from our family of Finsler metrics. Indeed, how to get a Finsler metric on an open disc in $\mathbf{H}^2$ that induces a given distance function on the boundary of that disk is a construction that is well known (see [2]). However, we want here to construct a family of Finsler metrics on an open disk in $\mathbf{H}^2$ that all induce on the boundary of that disk the same distance function as that induced by the hyperbolic Riemannian metric g_0 on $\mathbf{H}^2$, and that extend to g_0 outside the disk in a *smooth* way, this latter point being not something classical. Moreover, we expect the extended family $(F_\lambda)_{\lambda \in (-\varepsilon, \varepsilon)}$ to be also *smooth* with respect to the real parameter λ since this may be useful in studying the behaviour of some invariants associated with these Finsler metrics by differentiating them with respect to λ.

As corollaries to our Main Theorem, we get that *all* of the Riemannian rigidity results stated above *fail* to extend to the Finsler category for *surfaces*. In particular, Katok's rigidity result about closed hyperbolic surfaces (see Theorem 1.1, point (2)) does *not* hold any longer for Finsler metrics. By the way, it is interesting to note that Besson, Courtois and Gallot conjectured in their paper [3] (page 630) that Theorem 1.2 should remain true in the Finslerian context.

Corollary 1.1. *Let (S, g_0) be a closed hyperbolic surface. Then there exist a domain Ω in S with non-empty boundary $\partial\Omega$, a number $\varepsilon > 0$ and a continuous function $\Psi : (-\varepsilon, \varepsilon) \times TS \longrightarrow \mathbf{R}$ that is C^∞ on $(-\varepsilon, \varepsilon) \times (TS \smallsetminus \{0\})$ and such that for each $\lambda \in (-\varepsilon, \varepsilon)$, we have*

(1) $\mathcal{F}_\lambda(\cdot) := \Psi(\lambda, \cdot)$ *is a* smooth *strongly convex Finsler metric on* S,

(2) $\mathcal{F}_0$ *is associated with* g_0, *and* $\mathcal{F}_\lambda$ *is* not Riemannian *whenever* $\lambda \neq 0$,

(3) $d_{\mathcal{F}_\lambda} = d_{\mathcal{F}_0}$, *where* $d_{\mathcal{F}_0}$ *and* $d_{\mathcal{F}_\lambda}$ *are the metrics induced on* $\partial\Omega$ *by the distance functions on* S *associated respectively with* $\mathcal{F}_0$ *and* $\mathcal{F}_\lambda$.

Corollary 1.2. *Let (S, g_0) be a closed hyperbolic surface. Then there exist $\varepsilon > 0$ and a continuous function $\Psi : (-\varepsilon, \varepsilon) \times TS \longrightarrow \mathbf{R}$ that is C^∞ on $(-\varepsilon, \varepsilon) \times (TS \smallsetminus \{0\})$ and such that for each $\lambda \in (-\varepsilon, \varepsilon)$, we have*

(1) $\mathcal{F}_\lambda(\cdot) := \Psi(\lambda, \cdot)$ *is a* smooth *strongly convex Finsler metric on* S*,*

(2) $\mathcal{F}_0$ *is associated with* g_0*, and* $\mathcal{F}_\lambda$ *is* not Riemannian *whenever* $\lambda \neq 0$*,*

(3) the marked length spectrum of $\mathcal{F}_\lambda$ *is equal to that of* $\mathcal{F}_0$*.*

Corollary 1.3. *Let* (S, g_0) *be a closed hyperbolic surface. Then there exist* $\varepsilon > 0$ *and a continuous function* $\Psi : (-\varepsilon, \varepsilon) \times TS \longrightarrow \mathbf{R}$ *that is* C^∞ *on* $(-\varepsilon, \varepsilon) \times (TS \smallsetminus \{0\})$ *and such that for each* $\lambda \in (-\varepsilon, \varepsilon)$*, we have*

(1) $\mathcal{F}_\lambda(\cdot) := \Psi(\lambda, \cdot)$ *is a* smooth *strongly convex Finsler metric on* S*,*

(2) $\mathcal{F}_0$ *is associated with* g_0*, and* $\mathcal{F}_\lambda$ *is* not Riemannian *whenever* $\lambda \neq 0$*,*

(3) $h(\mathcal{F}_\lambda)^2 \operatorname{Vol}_{\mathcal{F}_\lambda}(S) = h(\mathcal{F}_0)^2 \operatorname{Vol}_{\mathcal{F}_0}(S)$*, where* Vol *denotes the Holmes-Thompson volume.*

2. Proofs of the Corollaries

In this section, we will explain how the Main Theorem implies Corollaries 1.1, 1.2 and 1.3.

Let $(\mathbf{H}^2, g_0)$ be the two-dimensional hyperbolic space and let F_0 be the (smooth strongly convex) Finsler metric associated with g_0. Let Γ be a discrete cocompact subgroup of g_0-isometries acting properly discontinuously on $\mathbf{H}^2$ without fixed points. Then $S = \mathbf{H}^2/\Gamma$ is a closed surface endowed with the quotient hyperbolic metric g_0 and the projection map $\pi : \mathbf{H}^2 \longrightarrow S$ is a Riemannian covering. Finally, let $\mathcal{D}$ be an open ball in $(\mathbf{H}^2, g_0)$ such that $\pi_{|\overline{\mathcal{D}}}$ is injective, where $\overline{\mathcal{D}}$ stands for the closure of $\mathcal{D}$ in $(\mathbf{H}^2, g_0)$.

Notations. For any Finsler metric $\mathcal{F}$ on S, we will denote by F the lift of $\mathcal{F}$ to $\mathbf{H}^2$ and by D_F the distance function on $\mathbf{H}^2$ associated with F. The $\mathcal{F}$-length (respectively F-length) of curves will be denoted by $L_{\mathcal{F}}$ (respectively L_F), and the induced metric on $\partial\mathcal{D}$ by D_F will be denoted by d_F. For $x \in \mathbf{H}^2$ and $R > 0$, let $B_F(x, R)$ denote the open ball about x in $\mathbf{H}^2$ of radius R with respect to D_F. In addition, given $x, y \in \mathbf{H}^2$, any D_F-distance minimizing geodesic $[0, 1] \longrightarrow \mathbf{H}^2$ connecting x to y will be denoted by $[x, y]_F$ (it is to be noticed that $\mathcal{F}$ is complete since S is closed, and thus F is complete too).

Corollary 1.1 will be a straightforward consequence of the following lemma:

Lemma 2.1. *There exist* $\varepsilon > 0$ *and a continuous function* $\Psi : (-\varepsilon, \varepsilon) \times TS \longrightarrow \mathbf{R}$ *that is* C^∞ *on* $(-\varepsilon, \varepsilon) \times (TS \smallsetminus \{0\})$ *and such that for each* $\lambda \in (-\varepsilon, \varepsilon)$*, we have*

(1) $\mathcal{F}_\lambda(\cdot) := \Psi(\lambda, \cdot)$ *is a smooth strongly convex Finsler metric on* S*,*

(2) $\mathcal{F}_0$ *is associated with* g_0*, and* $\mathcal{F}_\lambda$ *is not Riemannian whenever* $\lambda \neq 0$*,*

(3) $d_{F_\lambda} = d_{F_0}$*,*

(4) every two points in $\mathcal{D}$ *can be joined by a geodesic of* F_λ *whose image is contained in* $\mathcal{D}$*,*

(5) $F_\lambda(x, u) = F_0(x, u)$ *for all* $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ *and* $u \in T_x\mathbf{H}^2$*,*

(6) F_λ has no conjugate points.

Proof.

Roughly speaking, the proof will first consist in spreading out the family $(\mathsf{F}_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$ obtained in the Main Theorem all over $\mathbf{H}^2$ by the deck transformations of the universal covering $\pi : \mathbf{H}^2 \longrightarrow S$. Then, we will get a new family $(F_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$ of Finsler metrics on $\mathbf{H}^2$ that are invariant under the group Γ, which will make it possible to consider their quotients on the surface S. This will finally give rise to a family $(\mathcal{F}_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$ of Finsler metrics on S, each of them being equal to the Riemannian hyperbolic metric g_0 outside a small topological disk.

Let $\varepsilon > 0$ and $\Phi : (-\varepsilon, \varepsilon) \times T\mathbf{H}^2 \longrightarrow \mathbf{R}$ as given in the Main Theorem.

Since $\pi_{|\overline{\mathcal{D}}}$ is injective, $\overline{\mathcal{D}}$ is compact and $\pi : \mathbf{H}^2 \longrightarrow S$ is a covering map, there exists an open set U in $\mathbf{H}^2$ such that $\overline{\mathcal{D}} \subseteq U$ and $\pi_{|U}$ is still injective. So, as $\Phi(\lambda,(x,u)) = \mathsf{F}_0(x,u)$ for all $\lambda \in (-\varepsilon,\varepsilon)$, $x \in \mathbf{H}^2 \smallsetminus \mathcal{D}$ and $u \in T_x\mathbf{H}^2$ by property (5) in the Main Theorem, we can define the new continuous function $\widehat{\Phi} : (-\varepsilon,\varepsilon) \times T\mathbf{H}^2 \longrightarrow \mathbf{R}$ by setting $\widehat{\Phi}(\lambda,(\gamma(x),T_x\gamma\cdot u)) = \mathsf{F}_\lambda(x,u)$ for all $\gamma \in \Gamma$ if $x \in U$ and $u \in T_x\mathbf{H}^2$, and $\widehat{\Phi}(\lambda,(x,u)) = \mathsf{F}_0(x,u)$ if $x \in \mathbf{H}^2 \smallsetminus \Gamma(U)$ and $u \in T_x\mathbf{H}^2$.

Note that this definition makes sense since we have $\gamma(U) \cap U = \varnothing$ for all $\gamma \in \Gamma$ with $\gamma \neq \mathfrak{I}_{\mathbf{H}^2}$ (indeed, if $\gamma \in \Gamma$ and $x_0 \in U$ are such that $\gamma(x_0) \in U$, then necessarily $\gamma(x_0) = x_0$ by injectivity of $\pi_{|U}$, and hence $\gamma = \mathfrak{I}_{\mathbf{H}^2}$ since Γ has no fixed points).

This function $\widehat{\Phi}$ is then C^∞ on $(-\varepsilon,\varepsilon) \times (T\mathbf{H}^2 \smallsetminus \{0\})$ and for each $\lambda \in (-\varepsilon,\varepsilon)$ satisfies

(i) $\widehat{\Phi}(\lambda,\cdot)$ is a smooth strongly convex Finsler metric on $\mathbf{H}^2$,

(ii) $\widehat{\Phi}(0,\cdot)$ is associated with g_0, and $\widehat{\Phi}(\lambda,\cdot)$ is not Riemannian whenever $\lambda \neq 0$.

Since $\widehat{\Phi}(\lambda,(\gamma(x),T_x\gamma\cdot u)) = \widehat{\Phi}(\lambda,(x,u))$ for all $\lambda \in (-\varepsilon,\varepsilon)$, $(x,u) \in T\mathbf{H}^2$ and $\gamma \in \Gamma$ by construction, the quotient function $\mathit{\Psi} : (-\varepsilon,\varepsilon) \times TS \longrightarrow \mathbf{R}$ given by $\mathit{\Psi}(\lambda, T\pi(x,u)) = \widehat{\Phi}(\lambda,(x,u))$ is well defined and immediately satisfies points (1) and (2) of Lemma 2.1 thanks to points (i) and (ii) above. Furthermore, points (3) to (6) in the Main Theorem automatically yield points (3) to (6) of Lemma 2.1. $\square$

Proof of Corollary 1.1.

Choose $\Omega = \pi(\mathcal{D})$ and apply Lemma 2.1. $\square$

On the other hand, keeping in mind that d_{F_0} denotes the induced metric on $\partial\mathcal{D}$ by the distance function D_{F_0} on $\mathbf{H}^2$ associated with F_0, Corollaries 1.2 and 1.3 will need the following lemma:

Lemma 2.2. *Let $\mathcal{F}$ be a Finsler metric on S such that we have*

(1) $d_F = d_{F_0}$, and

(2) $F(x,u) = F_0(x,u)$ for all $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ and $u \in T_x\mathbf{H}^2$.

Then $D_F(x,y) = D_{F_0}(x,y)$ for all $x,y \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$.

Proof.

Fix $x, y \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ and let us consider a F-distance minimizing geodesic $[x, y]_F$ connecting x to y.

We will construct a curve σ also connecting x to y such that $L_{F_0}(\sigma) = L_F([x, y]_F)$, and hence conclude that $D_{F_0}(x, y) \leqslant L_{F_0}(\sigma) = L_F([x, y]_F) = D_F(x, y)$. A similar argument will give the reverse inequality, proving the lemma.

As the image of $[x, y]_F$ is compact, it intersects only a finite number $N \geqslant 0$ of connected components of the open set $\Gamma(\mathcal{D}) = \pi^{-1}(\pi(\mathcal{D}))$ in $\mathbf{H}^2$.

If $N = 0$, *i.e.*, if $[x, y]_F$ does not enter $\Gamma(\mathcal{D})$, then hypothesis (2) implies $L_F([x, y]_F) = L_{F_0}([x, y]_F)$, hence we may take $\sigma = [x, y]_F$ and obtain the result.

Suppose $N \geqslant 1$, and let C_1 be the first connected component of $\Gamma(\mathcal{D})$ that $[x, y]_F$ enters. Let e_1 be the point at which $[x, y]_F$ enters C_1 the first time, and o_1 be the point at which $[x, y]_F$ leaves C_1 the last time. Similarly let C_2 be the first connected component of $\Gamma(\mathcal{D})$ met by the geodesic $[o_1, y]_F$, define e_2 to be the point at which $[o_1, y]_F$ enters C_2 the first time, and o_2 to be the point at which $[o_1, y]_F$ leaves C_2 the last time. Continuing in this fashion, we define finite sequences $(C_i)_{1 \leqslant i \leqslant k}$, $(e_i)_{1 \leqslant i \leqslant k}$ and $(o_i)_{1 \leqslant i \leqslant k}$ by induction, where $k \in \{1, \ldots, N\}$ is such that the image of $[o_k, y]_F$ does not intersect $\Gamma(\mathcal{D})$ (see Figure 1).

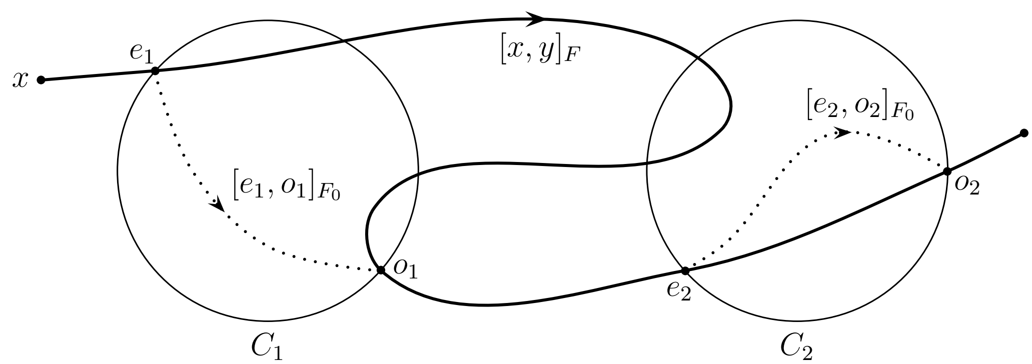

FIGURE 1. Proof of Lemma 2.2

Now we have

$$L_F([x, y]_F) = L_F([x, e_1]_F) + L_F([e_1, o_1]_F) + L_F([o_1, e_2]_F) + \cdots + L_F([e_k, o_k]_F) + L_F([o_k, y]_F).$$

The F-length and the F_0-length of each segment lying entirely outside $\Gamma(\mathcal{D})$ are already equal by hypothesis (2). In particular, we have

$$L_F([o_i, e_{i+1}]_F) = L_{F_0}([o_i, e_{i+1}]_F)$$

for each $1 \leqslant i \leqslant k-1$ (in case $k \geqslant 2$), as well as

$$L_F([x, e_1]_F) = L_{F_0}([x, e_1]_F)$$

and

$$L_F([o_k, y]_F) = L_{F_0}([o_k, y]_F).$$

Furthermore, for each $i \in \{1, \ldots, k\}$, observing that e_i and o_i lie in ∂C_i, we have

$$L_{F_0}([e_i, o_i]_{F_0}) = D_{F_0}(e_i, o_i) = d_{F_0}(e_i, o_i) = d_F(e_i, o_i) = D_F(e_i, o_i) = L_F([e_i, o_i]_F)$$

by hypothesis (1).

Thus, if σ is the curve

$$\sigma = [x, e_1]_F \# [e_1, o_1]_{F_0} \# [o_1, e_2]_F \# \cdots \# [e_{k-1}, o_{k-1}]_{F_0} \# [o_{k-1}, e_k]_F \# [e_k, o_k]_{F_0} \# [o_k, y]_F,$$

where $\#$ is the concatenation operator, then $L_{F_0}(\sigma) = L_F([x, y]_{\mathcal{F}})$, as desired.

The same argument reversing the roles of F_0 and F shows that $D_F(x, y) \leqslant D_{F_0}(x, y)$, and hence we have $D_F(x, y) = D_{F_0}(x, y)$, completing the proof. $\square$

Proof of Corollary 1.2.

Let $\varepsilon > 0$ and $\Psi : (-\varepsilon, \varepsilon) \times TS \longrightarrow \mathbf{R}$ as given by Lemma 2.1.

Fix $\lambda \in (-\varepsilon, \varepsilon)$ and consider a free homotopy class Σ of S that is *not* trivial. As in the proof of Lemma 2.1, let U be an open set in $\mathbf{H}^2$ such that $\overline{\mathcal{D}} \subseteq U$ and $\pi_{|U}$ is injective.

If $\sigma_\lambda : [0, 1] \longrightarrow S$ is a closed curve of shortest $\mathcal{F}_\lambda$-length within Σ, the image $\sigma_\lambda([0, 1])$ of σ_λ can not entirely lie in $\pi(\mathcal{D})$. Indeed, if this were the case, the image of the curve $f^{-1} \circ \sigma_\lambda$ would be included in $\mathcal{D}$, where $f : (U, F_\lambda) \longrightarrow (\pi(U), \mathcal{F}_\lambda)$ is the isometry induced by π and F_λ is the lift to $\mathbf{H}^2$ of $\mathcal{F}_\lambda$. But $f^{-1} \circ \sigma_\lambda$ is contractible to a point in $\mathcal{D}$ and thus σ_λ would be contractible to a point in $\pi(\mathcal{D})$, which is not possible since Σ is not trivial. So, there exists a point p_0 in the image of σ_λ that is not in $\pi(\mathcal{D})$.

Let $x_0 \in \mathbf{H}^2$ such that $\pi(x_0) = p_0$ and denote by $\overline{\sigma}_\lambda$ the unique lift of σ_λ to $\mathbf{H}^2$ starting at x_0.

If $y_0 \in \pi^{-1}(p_0)$ is the end point of $\overline{\sigma}_\lambda$, then we have $x_0, y_0 \in \mathbf{H}^2 \setminus \Gamma(\mathcal{D})$, and therefore

$$D_{F_\lambda}(x_0, y_0) = D_{F_0}(x_0, y_0) \tag{2.1}$$

by Lemma 2.2.

Now, we have

$$L_{F_\lambda}(\overline{\sigma}_\lambda) = L_{\mathcal{F}_\lambda}(\sigma_\lambda) \tag{2.2}$$

since $\pi : (\mathbf{H}^2, F_\lambda) \longrightarrow (S, \mathcal{F}_\lambda)$ is a local isometry. But this implies that $\overline{\sigma}_\lambda$ is a distance minimizing geodesic connecting x_0 to y_0 in $(\mathbf{H}^2, F_\lambda)$ because if this were not the case, there would be a F_λ-geodesic $[x_0, y_0]_{F_\lambda}$ such that $L_{F_\lambda}([x_0, y_0]_{F_\lambda}) < L_{F_\lambda}(\overline{\sigma}_\lambda)$ and hence $L_{F_\lambda}([x_0, y_0]_{F_\lambda}) < L_{\mathcal{F}_\lambda}(\sigma_\lambda)$ by Equation 2.2. Therefore, we would get $L_{\mathcal{F}_\lambda}(\pi \circ [x_0, y_0]_{F_\lambda}) < L_{\mathcal{F}_\lambda}(\sigma_\lambda)$, which is not possible since $\pi \circ [x_0, y_0]_{F_\lambda}$ is a closed curve that belongs to Σ (indeed, as $[x_0, y_0]_{F_\lambda}$ is homotopic to $\overline{\sigma}_\lambda$ in the simply connected space $\mathbf{H}^2$ with fixed ends x_0 and y_0, the closed curve $\pi \circ [x_0, y_0]_{F_\lambda}$ is homotopic to σ_λ in S with fixed base point p_0).

So, we have $L_{F_\lambda}(\overline{\sigma}_\lambda) = D_{F_\lambda}(x_0, y_0)$, and hence

$$L_{\mathcal{F}_\lambda}(\sigma_\lambda) = D_{F_0}(x_0, y_0) \tag{2.3}$$

by Equations 2.1 and 2.2.

Next, if $\overline{\sigma}_0 : [0,1] \longrightarrow \mathbf{H}^2$ is a D_{F_0}-distance minimizing geodesic connecting x_0 to y_0, the curve $\sigma_0 := \pi \circ \overline{\sigma}_0$ is a closed curve that belongs to Σ (same reasoning as above for $\pi \circ [x_0, y_0]_{F_\lambda}$) with $L_{F_0}(\overline{\sigma_0}) = L_{\mathcal{F}_0}(\sigma_0)$ since $\pi : (\mathbf{H}^2, F_0) \longrightarrow (S, \mathcal{F}_0)$ is a local isometry. Thus, we get

$$L_{\mathcal{F}_\lambda}(\sigma_\lambda) = L_{\mathcal{F}_0}(\sigma_0) \tag{2.4}$$

by Equation 2.3.

On the other hand, starting with is a closed curve $\tau_0 : [0,1] \longrightarrow S$ of shortest $\mathcal{F}_0$-length within Σ and using exactly the same steps as previously (reversing the roles of $\mathcal{F}_\lambda$ and $\mathcal{F}_0$), there exists a closed curve $\tau_\lambda : [0,1] \longrightarrow S$ that belongs to Σ and satisfies

$$L_{\mathcal{F}_0}(\tau_0) = L_{\mathcal{F}_\lambda}(\tau_\lambda). \tag{2.5}$$

Finally, Equations 2.4 and 2.5 yield

$$L_{\mathcal{F}_\lambda}(\sigma_\lambda) \leqslant L_{\mathcal{F}_\lambda}(\tau_\lambda) = L_{\mathcal{F}_0}(\tau_0) \leqslant L_{\mathcal{F}_0}(\sigma_0) = L_{\mathcal{F}_\lambda}(\sigma_\lambda),$$

and therefore $L_{\mathcal{F}_\lambda}(\sigma_\lambda) = L_{\mathcal{F}_0}(\tau_0)$.

This proves Corollary 1.2. □

Remark. It is to be noticed here that the proof of Corollary 1.2 shows that for each non-trivial free homotopy class Σ for S and each $\lambda \in (-\lambda_0, \lambda_0)$, there is a *unique* (up to reparameterization) closed curve $\mathbf{T}^1 := \mathbf{R}/\mathbf{Z} \longrightarrow S$ in Σ of shortest $\mathcal{F}_\lambda$-length.

Proof.

Let $\sigma : \mathbf{T}^1 \longrightarrow S$ and $\tau : \mathbf{T}^1 \longrightarrow S$ be closed curves in Σ of (the same) shortest $\mathcal{F}_\lambda$-length (thus $\mathcal{F}_\lambda$-geodesics) and prove they are equal up to a translation in $\mathbf{T}^1$.

If there were $p_0 \in \sigma(\mathbf{T}^1) \smallsetminus \pi(\mathcal{D})$ and $p_1 \in \tau(\mathbf{T}^1) \smallsetminus \pi(\mathcal{D})$ with $p_0 \notin \tau(\mathbf{T}^1)$ and $p_1 \notin \sigma(\mathbf{T}^1)$, then the same reasoning as in the proof of Corollary 1.2 would lead to the existence of closed curves $\sigma_0 : \mathbf{T}^1 \longrightarrow S$ and $\tau_0 : \mathbf{T}^1 \longrightarrow S$ in Σ of shortest $\mathcal{F}_0$-length with $p_0 \in \sigma_0(\mathbf{T}^1)$ and $p_1 \in \tau_0(\mathbf{T}^1)$. As $p_0 \neq p_1$, we would get $\sigma_0 \neq \tau_0$, which is not possible since it is well known there is a unique (up to reparameterization) closed curve $\mathbf{T}^1 \longrightarrow S$ in Σ of shortest $\mathcal{F}_0$-length since $\mathcal{F}_0$ is Riemannian hyperbolic.

So, we have $\sigma(\mathbf{T}^1) \smallsetminus \pi(\mathcal{D}) \subseteq \tau(\mathbf{T}^1)$ or $\tau(\mathbf{T}^1) \smallsetminus \pi(\mathcal{D}) \subseteq \sigma(\mathbf{T}^1)$, and this implies there exist $t_0 \in \mathbf{T}^1$ and a neighborhood $\mathcal{V}$ of 0 in $\mathbf{T}^1$ such that $\sigma(t) = \tau(t + t_0)$ for all $t \in \mathcal{V}$. Thus $\sigma(t) = \tau(t + t_0)$ for all $t \in \mathbf{T}^1$ since σ and $\mathbf{T}^1 \ni t \longmapsto \tau(t + t_0) \in S$ are $\mathcal{F}_\lambda$-geodesics. □

Last, we discuss the volume growth entropy, considered with respect to the Holmes-Thompson volume, which we now define. Let $(M, \mathcal{F})$ be an n-dimensional Finsler manifold. For each $p \in M$, let $B_p^{\mathcal{F}} M := \{v \in T_pM \mid \mathcal{F}(p, v) < 1\}$ be the unit open ball in T_pM for the norm

$\mathcal{F}(p,\cdot)$, and $\left(B^{\mathcal{F}}_p M\right)^\circ$ its dual set in $T^*_p M$ (recall that for any set X in a finite dimensional vector space V, we have $X^\circ := \{\varphi \in V^* \mid \varphi(v) \leqslant 1 \text{ for all } v \in X\} \subseteq V^*$). Then define the unit $\mathcal{F}$-ball co-tangent bundle

$$\begin{aligned}\left(B^{\mathcal{F}} M\right)^\circ &:= \bigcup_{p\in M} \{p\} \times \left(B^{\mathcal{F}}_p M\right)^\circ \\ &= \left\{(p,\varphi) \in T^*M \mid \varphi(v) \leqslant 1 \ \text{ for all } \ v \in B^{\mathcal{F}}_p M\right\} \subseteq T^*M.\end{aligned}$$

Let ω be the canonical symplectic form on T^*M given by $\omega := \mathrm{d}\alpha$, where α is the Liouville 1-form on T^*M, and

$$\Omega := \frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$$

be the canonical volume form on T^*M.

For any Borel subset $A \subseteq M$, define the Holmes-Thompson volume of A by

$$\mathrm{Vol}_{\mathcal{F}}(A) := \frac{1}{C_n} \int_{(B^{\mathcal{F}} M)^\circ_{|A}} \Omega,$$

where $\left(B^{\mathcal{F}} M\right)^\circ_{|A} := \left\{(p,\varphi) \in \left(B^{\mathcal{F}} M\right)^\circ \mid p \in A\right\} \subseteq T^*M$ and C_n is the volume of the unit open ball in n-dimensional Euclidean space.

The Holmes-Thompson volume generalizes the Riemannian volume in the sense that if $\mathcal{F}$ is the Finsler metric associated with a Riemannian metric g on M, then $\mathrm{Vol}_{\mathcal{F}} = \mathrm{vol}_g$. Note that for a Finsler manifold there is another choice of volume that generalizes Riemannian volume called the Busemann volume which corresponds to the Hausdorff measure (see [6], page 192). It is to be mentionned that partial results concerning the entropy rigidity question have been obtained using the Busemann volume (see [18] and [5]). Moreover, to get a taste of the difference between these two notions of volume in Finsler geometry, one may have a look at [1].

Remark 2.1.

(1) If $\mathcal{F}^* : T^*M \longrightarrow \mathbf{R}$ is the dual Finsler metric of $\mathcal{F}$ defined by

$$\mathcal{F}^*(p,\varphi) := \max\{\varphi(v) \mid v \in T_pM \ \text{ and } \ \mathcal{F}(p,v) = 1\},$$

then, for each $p \in M$, we have

$$\left(B^{\mathcal{F}}_p M\right)^\circ = B^{\mathcal{F}^*}_p M := \{\varphi \in T^*_p M \mid \mathcal{F}^*(p,\varphi) < 1\} \subseteq T^*_p M.$$

(2) Given any Riemannian metric g on M, we have the formula

$$\mathrm{Vol}_{\mathcal{F}}(A) = \frac{1}{C_n} \int_A \mathrm{vol}_{g^*(p)}\big(\left(B^{\mathcal{F}}_p M\right)^\circ\big) \mathrm{d}\mathrm{vol}_g(p)$$

for any Borel subset $A \subseteq M$, where $g^*(p)$ is the dual scalar product of $g(p)$ on $T^*_p M$ and $\mathrm{vol}_{g^*(p)}$ its associated Haar measure (see [7]).

(3) In case the Finsler metric $\mathcal{F}$ is *smooth* and *strongly convex*, let $\Omega_{\mathcal{F}}$ be the symplectic volume form on $TM \setminus \{0\}$ associated with $\mathcal{F}$ defined as the pullback of the canonical volume form on T^*M by the Legendre transform $TM \setminus \{0\} \longrightarrow T^*M$ induced by $\mathcal{F}$ (this map is a local diffeomorphism since $\mathcal{F}$ is strongly convex).
Then we can write
$$\mathrm{Vol}_{\mathcal{F}}(A) = \frac{1}{C_n} \int_{(TM\setminus\{0\})\cap B^{\mathcal{F}}M_{|A}} \Omega_{\mathcal{F}},$$
where $B^{\mathcal{F}}M_{|A} := \{(p,v) \in TM \mid p \in A \text{ and } \mathcal{F}(p,v) < 1\} \subseteq TM$.

We shall now use the Main Theorem together with the following key result by Ivanov to prove Corollary 1.3.

Theorem 2.1 (Ivanov, [15])**.** *Let Δ be an open Euclidean disk in $\mathbf{R}^2$, and consider* smooth strongly convex *Finsler metrics $\mathfrak{F}_0$ and $\mathfrak{F}$ on $\mathbf{R}^2$. Assume that* every *two points in Δ can be joined* within $\overline{\Delta}$ *by a* unique *(up to reparametrization) geodesic of $\mathfrak{F}_0$ and by a geodesic of $\mathfrak{F}$. Then, if $d_{\mathfrak{F}}(x,y) \geqslant d_{\mathfrak{F}_0}(x,y)$ for all $x, y \in \partial\Delta$, we have $\mathrm{Vol}_{\mathfrak{F}}(\Delta) \geqslant \mathrm{Vol}_{\mathfrak{F}_0}(\Delta)$, where $d_{\mathfrak{F}_0}$ and $d_{\mathfrak{F}}$ are the metrics induced on $\partial\Delta$ by the distance functions on $\mathbf{R}^2$ associated respectively with $\mathfrak{F}_0$ and $\mathfrak{F}$.*

From this, we get

Consequence 2.1. *Let $\mathfrak{F}$ be a smooth strongly convex Finsler metric on $\mathbf{H}^2$ without conjugate points and such that every two points in $\mathcal{D}$ can be joined by a geodesic of $\mathfrak{F}$ whose image is contained in $\mathcal{D}$. Then, if $d_{\mathfrak{F}} = d_{F_0}$, we have $\mathrm{Vol}_{\mathfrak{F}}(\mathcal{D}) = \mathrm{Vol}_{F_0}(\mathcal{D})$.*

Proof.
First of all, the hyperbolic Finsler metric F_0 is smooth and strongly convex, and every two points in $\mathbf{H}^2$ can be joined by a unique (up to reparametrization) geodesic of F_0. Therefore, since $\mathcal{D}$ is an open ball in $(\mathbf{H}^2, F_0)$, the unique (up to reparametrization) F_0-geodesic joining two points in $\mathcal{D}$ has its image contained in $\mathcal{D}$. Thus, $d_{\mathfrak{F}} \geqslant d_{F_0}$ yields $\mathrm{Vol}_{\mathfrak{F}}(\mathcal{D}) \geqslant \mathrm{Vol}_{F_0}(\mathcal{D})$ by Theorem 2.1.

On the other hand, since the smooth strongly convex Finsler metric $\mathfrak{F}$ has no conjugate points, every two points in $\mathcal{D}$ can *actually* be joined within $\mathcal{D}$ by a *unique* (up to reparametrization) geodesic of $\mathfrak{F}$. So, using $d_{\mathfrak{F}} \leqslant d_{F_0}$, we have $\mathrm{Vol}_{\mathfrak{F}}(\mathcal{D}) \leqslant \mathrm{Vol}_{F_0}(\mathcal{D})$ still by Theorem 2.1.

Conclusion: $\mathrm{Vol}_{\mathfrak{F}}(\mathcal{D}) = \mathrm{Vol}_{F_0}(\mathcal{D})$. $\square$

Let us now prove the following two independent lemmas:

Lemma 2.3. *Let $\mathcal{F}$ be a Finsler metric on S such that*

(1) $d_F = d_{F_0}$,
(2) $F(x,u) = F_0(x,u)$ for all $x \in \mathbf{H}^2 \setminus \Gamma(\mathcal{D})$ and $u \in T_x\mathbf{H}^2$.

Let $c > 0$ be a constant such that $\operatorname{diam}_F(\mathcal{D}) \leqslant c$ *and* $\operatorname{diam}_{F_0}(\mathcal{D}) \leqslant c$*. Then for all* $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ *and $R > c$, we have*

$$B_{F_0}(x, R-c) \subseteq B_F(x,R) \subseteq B_{F_0}(x, R+c).$$

and

Lemma 2.4. *Let $\mathcal{F}$ be a Finsler metric on S such that*

(1) $\operatorname{Vol}_F(\mathcal{D}) = \operatorname{Vol}_{F_0}(\mathcal{D})$*,*

(2) $F(x,u) = F_0(x,u)$ *for all* $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ *and* $u \in T_x\mathbf{H}^2$*.*

Let $c > 0$ be a constant such that $\operatorname{diam}_F(\mathcal{D}) \leqslant c$ *and* $\operatorname{diam}_{F_0}(\mathcal{D}) \leqslant c$*. Then for all* $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ *and $R > 2c$, we have*

$$\operatorname{Vol}_F(B_{F_0}(x, R-c)) \leqslant \operatorname{Vol}_{F_0}(B_{F_0}(x,R)) \leqslant \operatorname{Vol}_F(B_{F_0}(x, R+c))\,.$$

Proof of Lemma 2.3.

Let $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ and $y \in \mathbf{H}^2$.

We will show

$$D_{F_0}(x,y) - c \leqslant D_F(x,y) \leqslant D_{F_0}(x,y) + c,$$

which immediately implies the result.

Let us fix distance minimizing geodesics $[x,y]_F$ and $[x,y]_{F_0}$ for F and F_0 respectively connecting x to y.

By Lemma 2.2, we know that if $y \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$, then $D_F(x,y) = D_{F_0}(x,y)$, and the inequalities above hold.

So, suppose y is in a connected component C of $\Gamma(\mathcal{D})$ and let e_{F_0} and e_F be the points at which respectively $[x,y]_{F_0}$ and $[x,y]_F$ enter C the first time (e_{F_0} and e_F lie in ∂C, see Figure 2).

As $\operatorname{diam}_{F_0}(\mathcal{D}) \leqslant c$ and $\operatorname{diam}_F(\mathcal{D}) \leqslant c$, we have $D_{F_0}(e_{F_0}, y) \leqslant c$ and $D_F(e_F, y) \leqslant c$. Furthermore, by Lemma 2.2 and since x, e_{F_0} and e_F all lie outside $\Gamma(\mathcal{D})$, we have $D_F(x, e_F) = D_{F_0}(x, e_F)$ and $D_F(x, e_{F_0}) = D_{F_0}(x, e_{F_0})$. Thus

$$D_F(x,y) \leqslant D_F(x,e_F) + D_F(e_F,y) \leqslant D_{F_0}(x,e_F) + c$$

and

$$D_{F_0}(x,y) \leqslant D_{F_0}(x,e_{F_0}) + D_{F_0}(e_{F_0},y) \leqslant D_F(x,e_{F_0}) + c.$$

We conclude that

$$D_{F_0}(x,y) - c \leqslant D_F(x,y) \leqslant D_{F_0}(x,y) + c,$$

as claimed. □

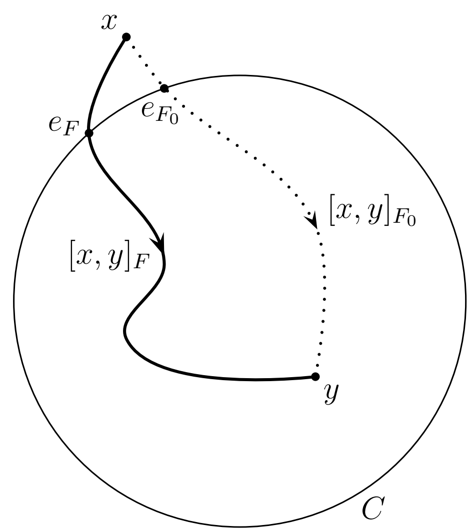

FIGURE 2. Proof of Lemma 2.3

Proof of Lemma 2.4.
Let $x \in \mathbf{H}^2$ and $R > 2c$.

By assumption (1), we have $\mathrm{Vol}_F(\gamma(\mathcal{D})) = \mathrm{Vol}_{F_0}(\gamma(\mathcal{D}))$ for each $\gamma \in \Gamma$ since Γ is a group of isometries for both F and F_0.

On the other hand, by assumption (2), Borel sets in $\mathbf{H}^2$ not intersecting $\Gamma(\mathcal{D})$ have the same Holmes-Thompson volume with respect to F_0 as with respect to F (the boundary of $\Gamma(\mathcal{D})$ being a set of zero measure for both Holmes-Thompson volumes).

Let $\mathcal{U}$ be the union of the connected components of $\Gamma(\mathcal{D})$ that intersect $\partial B_{F_0}(x, R)$. Then

$$\mathrm{Vol}_F(B_{F_0}(x, R) \smallsetminus \mathcal{U}) = \mathrm{Vol}_{F_0}(B_{F_0}(x, R) \smallsetminus \mathcal{U})$$

and

$$\mathrm{Vol}_F(B_{F_0}(x, R) \cup \mathcal{U}) = \mathrm{Vol}_{F_0}(B_{F_0}(x, R) \cup \mathcal{U}) \,.$$

Since for each connected component $\gamma(\mathcal{D})$ ($\gamma \in \Gamma$) of $\Gamma(\mathcal{D})$ we have $\mathrm{diam}_{F_0}(\gamma(\mathcal{D})) \leqslant c$ and $\mathrm{diam}_F(\gamma(\mathcal{D})) \leqslant c$, one gets

$$B_{F_0}(x, R - c) \subseteq B_{F_0}(x, R) \smallsetminus \mathcal{U} \subseteq B_{F_0}(x, R) \subseteq B_{F_0}(x, R) \cup \mathcal{U} \subseteq B_{F_0}(x, R + c),$$

and hence

$$\mathrm{Vol}_F(B_{F_0}(x, R - c)) \leqslant \mathrm{Vol}_{F_0}(B_{F_0}(x, R)) \leqslant \mathrm{Vol}_F(B_{F_0}(x, R + c)) \,.$$

□

Before proving Corollary 1.3, we need the following lemma about the volume growth entropy:

Lemma 2.5. *Let $\mathcal{F}$ be a Finsler metric on S such that*

(1) $d_F = d_{F_0}$,

(2) $\mathrm{Vol}_F(\mathcal{D}) = \mathrm{Vol}_{F_0}(\mathcal{D})$, *and*

(3) $F(x,u) = F_0(x,u)$ *for all* $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ *and* $u \in T_x\mathbf{H}^2$.

Then $h(F) = h(F_0)$.

Proof.

Choose $x \in \mathbf{H}^2 \smallsetminus \Gamma(\mathcal{D})$ and $R > 2c$.

By Lemma 2.3, we have

$$B_{F_0}(x, R-c) \subseteq B_F(x,R) \subseteq B_{F_0}(x, R+c).$$

Therefore

$$\mathrm{Vol}_F(B_{F_0}(x,R-c)) \leqslant \mathrm{Vol}_F(B_F(x,R)) \leqslant \mathrm{Vol}_F(B_{F_0}(x,R+c))\,,$$

and hence, by Lemma 2.4,

$$\mathrm{Vol}_{F_0}(B_{F_0}(x,R-2c)) \leqslant \mathrm{Vol}_F(B_F(x,R)) \leqslant \mathrm{Vol}_{F_0}(B_{F_0}(x,R+2c))\,.$$

But

$$h(F_0) = \lim_{R\to+\infty} \frac{1}{R} \log\left(\mathrm{Vol}_{F_0}(B_{F_0}(x,R-2c))\right) = \lim_{R\to+\infty} \frac{1}{R} \log\left(\mathrm{Vol}_{F_0}(B_{F_0}(x,R+2c))\right).$$

Thus $h(F) = h(F_0)$, as desired. □

We are now able to prove Corollary 1.3.

Proof of Corollary 1.3.

Let $\varepsilon > 0$ and $\Psi : (-\varepsilon, \varepsilon) \times TS \longrightarrow \mathbf{R}$ as given by Lemma 2.1.

Fixing $\lambda \in (-\varepsilon, \varepsilon)$, the smooth strongly convex Finsler metric F_λ on $\mathbf{H}^2$ satisfies points (4) and (6) of Lemma 2.1, and thus the hypotheses of Consequence 2.1. Furthermore, as it satisfies point (3) in Lemma 2.1, we then get $\mathrm{Vol}_{F_\lambda}(\mathcal{D}) = \mathrm{Vol}_{F_0}(\mathcal{D})$ by Consequence 2.1.

So, by point (5) in Lemma 2.1, F_λ satisfies all the three hypotheses of Lemma 2.5. Therefore, according to this latter lemma, $h(F_\lambda) = h(F_0)$, or equivalently $h(\mathcal{F}_\lambda) = h(\mathcal{F}_0)$.

Now, let U be an open set in $\mathbf{H}^2$ such that $\overline{\mathcal{D}} \subseteq U$ and $\pi_{|U}$ is injective as in the proof of Lemma 2.1. Then π induces isometries from (U, F_λ) onto $(\pi(U), \mathcal{F}_\lambda)$ and from (U, F_0) onto $(\pi(U), \mathcal{F}_0)$, which yield $\mathrm{Vol}_{F_\lambda}(\mathcal{D}) = \mathrm{Vol}_{\mathcal{F}_\lambda}(\pi(\mathcal{D}))$ and $\mathrm{Vol}_{F_0}(\mathcal{D}) = \mathrm{Vol}_{\mathcal{F}_0}(\pi(\mathcal{D}))$.

Hence $\mathrm{Vol}_{\mathcal{F}_\lambda}(\pi(\mathcal{D})) = \mathrm{Vol}_{\mathcal{F}_0}(\pi(\mathcal{D}))$.

On the other hand, point (5) in Lemma 2.1 implies that $\mathcal{F}_0(p,v) = \mathcal{F}_0(p,v)$ for all $p \in S \smallsetminus \pi(\mathcal{D})$ and $v \in T_pS$. Thus $\mathrm{Vol}_{\mathcal{F}_\lambda}(S \smallsetminus \pi(\mathcal{D})) = \mathrm{Vol}_{\mathcal{F}_0}(S \smallsetminus \pi(\mathcal{D}))$, and finally $\mathrm{Vol}_{\mathcal{F}_\lambda}(S) = \mathrm{Vol}_{\mathcal{F}_0}(S)$.

Conclusion: $h(\mathcal{F}_\lambda)^2\,\mathrm{Vol}_{\mathcal{F}_\lambda}(S) = h(\mathcal{F}_0)^2\,\mathrm{Vol}_{\mathcal{F}_0}(S)$.

This finishes the proof of Corollary 1.3. □

3. Proof of the Main Theorem

Throughout all this section, denote by $\langle\cdot,\cdot\rangle$ the usual scalar product in $\mathbf{R}^2$ and $|\cdot|$ its associated norm. Let $\mathbf{H}^2 := \{p \in \mathbf{R}^2 \mid |p| < 1\} \subseteq \mathbf{R}^2$ endowed with the Klein metric g_0 that is given by

$$g_0(p)\cdot(v,w) := \frac{\langle v,w\rangle}{1-|p|^2} + \frac{\langle p,v\rangle\,\langle p,w\rangle}{(1-|p|^2)^2} \tag{3.1}$$

for all $p \in \mathbf{H}^2$ and $v, w \in T_p\mathbf{H}^2 = \mathbf{R}^2$.

Thus $(\mathbf{H}^2, g_0)$ is a model of the hyperbolic plane where images of the geodesics are *affine segments.*

For each $r \in (0,1]$, let

$$\mathcal{D}(r) := \{p \in \mathbf{R}^2 \mid |p| < r\} \subseteq \mathbf{H}^2.$$

Finally, fix an arbitrary $R \in (0,1)$, let $\mathcal{D} := \mathcal{D}(R)$, and denote by d_{g_0} the induced metric on $\partial\mathcal{D}$ by the distance function on $\mathbf{H}^2$ associated with g_0.

3.1. Arcostanzo's construction.

In [2], Arcostanzo gives conditions on a distance d on $\partial\mathcal{D}$ and a set $\mathfrak{S}$ of parameterized curves $\gamma : [0,1] \longrightarrow \overline{\mathcal{D}}$ in such a way that there exists a Finsler metric F on $\mathcal{D}$ whose associated distance on $\mathcal{D}$ extends to a distance on $\overline{\mathcal{D}}$ that induces the metric d on $\partial\mathcal{D}$ and such that $\{\gamma_{|(0,1)} \mid \gamma \in \mathfrak{S}\}$ coincides with the set of maximal geodesics of $\mathcal{F}$ after reparametrization by $(0,1)$. We will state this result precisely in the specific case when the distance on $\partial\mathcal{D}$ is d_{g_0}, though more general results are established in [2].

We begin by giving Arcostanzo's conditions on a set of parameterized curves.

Definition 3.1. A set $\mathfrak{S}$ of parameterized curves $\gamma : [0,1] \longrightarrow \mathbf{R}^2$ is said to be admissible for $\mathcal{D}$ if and only if the following properties hold:

1. each $\gamma \in \mathfrak{S}$ is C^∞, regular, injective, and satisfies $\gamma((0,1)) \subseteq \mathcal{D}$;
2. for each $\gamma \in \mathfrak{S}$, we have $\gamma(0), \gamma(1) \in \partial\mathcal{D}$;
3. for any $p, q \in \overline{\mathcal{D}}$ with $p \neq q$, there exists a unique $(\gamma, t_0, t_1) \in \mathfrak{S} \times [0,1] \times [0,1]$ such that $p = \gamma(t_0)$ and $q = \gamma(t_1)$ with $t_0 < t_1$;
4. for any $p \in \mathcal{D}$ and $v \in T_p\mathcal{D} = \mathbf{R}^2$ with $v \neq 0$, there exists a unique $(\gamma, t) \in \mathfrak{S} \times (0,1)$ such that $p = \gamma(t)$ and $\gamma'(t)$ is parallel to v with the same direction.

For Arcostanzo's construction to yield a Finsler metric and not just a distance on $\mathcal{D}$, a certain amount of regularity is required about the way the end points $\gamma(0), \gamma(1) \in \partial\mathcal{D}$ depend on the parameterized curve $\gamma \in \mathfrak{S}$.

More precisely, given $\mathfrak{S}$ an admissible set of parameterized curves for $\mathcal{D}$, for each $x \in \partial\mathcal{D}$ and $p \in \mathcal{D}$, there is a unique $\gamma \in \mathfrak{S}$ such that $x = \gamma(0)$ and $p \in \gamma([0,1])$ according to point (3)

in Definition 3.1. Setting $\sigma(x,p) := \gamma(1)$, we then get a map $\sigma : \partial\mathcal{D} \times \mathcal{D} \longrightarrow \partial\mathcal{D}$ we will call the 'end point map' associated with $\mathfrak{S}$ (see Figure 3).

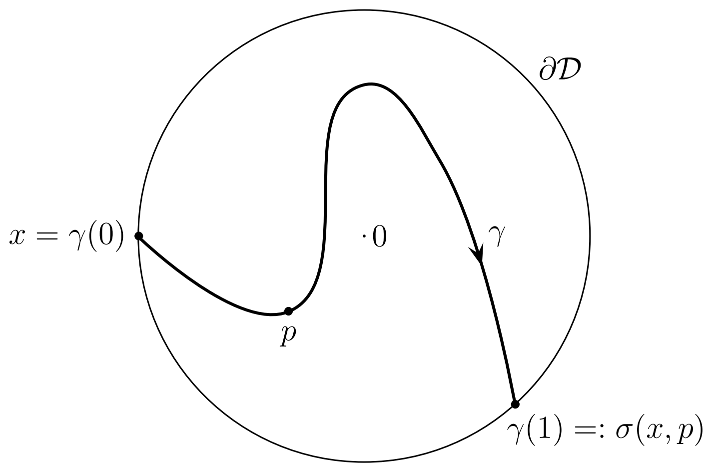

FIGURE 3. The 'end point map' σ

Remark. For any $x, y \in \partial\mathcal{D}$ and $p \in \mathcal{D}$, we obviously have: $\sigma(x,p) = y \iff \sigma(y,p) = x$.

Definition 3.2. An admissible set of parameterized curves $\mathfrak{S}$ for $\mathcal{D}$ satisfies Arcostanzo's Property (C) if and only if

(1) the associated 'end point map' $\sigma : \partial\mathcal{D} \times \mathcal{D} \longrightarrow \partial\mathcal{D}$ is C^∞, and

(2) for every $(x,p) \in \partial\mathcal{D} \times \mathcal{D}$ and every $v \in T_p\mathcal{D} = \mathbf{R}^2$ with $v \neq 0$, we have the equivalence

$$\frac{\partial\sigma}{\partial p}(x,p)\cdot v = 0 \iff v \text{ and } \gamma'(t) \text{ are parallel vectors,}$$

where (γ, t) is the unique element in $\mathfrak{S} \times (0,1)$ such that $x = \gamma(0)$ and $p = \gamma(t)$ according to point (3) in Definition 3.1 (with $t_0 = 0$ and $t_1 = t$).

Remark 3.1. Point (2) in Definition 3.2 can be reformulated in another way:

For every $(x,p) \in \partial\mathcal{D} \times \mathcal{D}$ and every $v \in T_p\mathcal{D} = \mathbf{R}^2$ with $v \neq 0$, we have the equivalence

$$\frac{\partial\sigma}{\partial p}(x,p)\cdot v = 0 \iff \left(x = e^-(p,v) \quad \text{or} \quad x = e^+(p,v)\right),$$

where $e^-(p,v) := \gamma(0)$ and $e^+(p,v) := \gamma(1)$ if γ denotes the unique parameterized curve in $\mathfrak{S}$ given by point (4) in Definition 3.1.

Arcostanzo points out, for example, that the set of maximal geodesics in $\overline{\mathcal{D}}$ (reparameterized by $[0,1]$) of a negatively curved Riemannian metric on an open neighborhood of $\overline{\mathcal{D}}$ is admissible for $\mathcal{D}$ and satisfies Property (C) (see [2], page 242).

We can now state Arcostanzo's result:

Theorem 3.1 ([2], Théorème 2, page 243)**.** *Let $\mathfrak{S}$ be an admissible set of parameterized curves for $\mathcal{D}$ that satisfies Property (C). Then there exists a* unique *Finsler metric F on $\mathcal{D}$ whose associated distance function on $\mathcal{D}$ extends to a distance on $\overline{\mathcal{D}}$ that induces the metric d_{g_0} on $\partial\mathcal{D}$ and such that $\{\gamma_{|(0,1)} \mid \gamma \in \mathfrak{S}\}$ coincides with the set of maximal geodesics of F after reparametrization by $(0,1)$. Its precise formula is given by*

$$\mathsf{F}(p,v) = \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(x, \sigma(x,p)) \left| \frac{\partial \sigma}{\partial p}(x,p) \cdot v \right| \mathrm{d}x$$

for any $(p,v) \in T\mathcal{D} = \mathcal{D} \times \mathbf{R}^2$, where $|\cdot|$ stands for the canonical Euclidean norm on $\mathbf{R}^2$ and $\mathrm{d}x$ denotes the canonical measure on the Euclidean circle $\partial\mathcal{D} = R\mathbf{S}^1$.
Moreover, this Finsler metric is smooth.

Remark 3.2.

(1) Given $x_0 = Re^{it_0}$ and $y_0 = Re^{is_0}$ in $\partial\mathcal{D} = R\mathbf{S}^1$ such that $x_0 \neq y_0$, the partial derivative $\dfrac{\partial^2 d_{g_0}}{\partial x \partial y}(x_0, y_0)$ is defined as to be equal to $\dfrac{\partial^2 f}{\partial t \partial s}(t_0, s_0) \in \mathbf{R}$, where $f(t,s) = d_{g_0}(Re^{it}, Re^{is})$ for all $t, s \in \mathbf{R}$.

(2) Arcostanzo's result applies here since g_0 is the hyperbolic metric and an easy computation then shows that $\dfrac{\partial^2 d_{g_0}}{\partial x \partial y}(x,y) > 0$ for any $x, y \in \partial\mathcal{D}$ with $x \neq y$.

(3) By uniqueness of F in Theorem 3.1, if we choose $\mathfrak{S}$ to be the set of maximal geodesics of g_0 in $\overline{\mathcal{D}}$ after reparametrization by $[0,1]$ (whose images are the chords of the Euclidean circle $\partial\mathcal{D}$), then we get that F equals the restriction to $T\mathcal{D}$ of the Finsler metric F_0 on $\mathbf{H}^2$ associated with g_0.

(4) If we choose $\mathfrak{S}$ to be the set of maximal geodesics in $\overline{\mathcal{D}}$ (reparameterized by $[0,1]$) of a negatively curved Riemannian metric on an open neighborhood of $\overline{\mathcal{D}}$ (the set $\mathfrak{S}$ is then admissible for $\mathcal{D}$ and satisfies Property (C) as shown by Arcostanzo in [2], page 242), the unique Finsler metric on $\mathcal{D}$ we get by Theorem 3.1 is *not* Riemannian.

(5) The existence of a unique Finsler metric F on $\mathcal{D}$ given by the formula in Theorem 3.1 still holds without the assumption that the admissible set of parameterized curves $\mathfrak{S}$ for $\mathcal{D}$ has Property (C), but in that case F is *not necessarily* reversible nor smooth.

Remark 3.3. Although it is not written in [2], the fact that F is a *smooth* Finsler metric on $\mathcal{D}$ can be proved as follows.

Proof.

Consider the map $\Upsilon : \mathcal{D} \times \mathbf{R}^2 \times \partial\mathcal{D} \longrightarrow \mathbf{R}^2$ defined by

$$\Upsilon((p,v),x) := \frac{1}{4}\frac{\partial^2 d_{g_0}}{\partial x \partial y}(x,\sigma(x,p))\frac{\partial \sigma}{\partial p}(x,p)\cdot v.$$

Since d_{g_0} is C^∞ on $(\partial\mathcal{D} \times \partial\mathcal{D}) \setminus \{(x,x) \mid x \in \partial\mathcal{D}\}$ and σ is C^∞ on $\partial\mathcal{D} \times \mathcal{D}$ (point (1) in Definition 3.2) which satisfies $\sigma(x,p) \neq x$ for all $(x,p) \in \partial\mathcal{D} \times \mathcal{D}$, the positive function $(x,p) \mapsto \dfrac{\partial^2 d_{g_0}}{\partial x \partial y}(x,\sigma(x,p))$ is C^∞ on $\partial\mathcal{D} \times \mathcal{D}$, and therefore Υ is C^∞.

This implies in particular that Υ is continuous, thus the function $\mathsf{F} : T\mathcal{D} = \mathcal{D} \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ in Theorem 3.1 given by $\mathsf{F}(p,v) = \displaystyle\int_{\partial\mathcal{D}} |\Upsilon((p,v),x)| \mathrm{d}x$ is well defined and continuous.

On the other hand, for any $((p,v),x) \in \mathcal{D} \times (\mathbf{R}^2 \setminus \{0\}) \times \partial\mathcal{D}$, the vector $\dfrac{\partial \sigma}{\partial p}(x,p)\cdot v \in \mathbf{R}^2$ vanishes iff $x = e^-(p,v)$ or $x = e^+(p,v)$ (point (2) in Definition 3.2). So, given $(p,v) \in \mathcal{D} \times (\mathbf{R}^2 \setminus \{0\})$, the differential $\dfrac{\partial|\Upsilon|}{\partial(p,v)}((p,v),x) \in \mathrm{L}(\mathbf{R}^4, \mathbf{R})$ exists for all $x \in \partial\mathcal{D} \setminus \{e^-(p,v), e^+(p,v)\}$ and writes

$$\begin{aligned}
\frac{\partial|\Upsilon|}{\partial(p,v)}((p,v),x)\cdot(w,\xi) &= \frac{\partial|\Upsilon|}{\partial p}((p,v),x)\cdot w + \frac{\partial \Upsilon}{\partial v}((p,v),x)\cdot\xi \\
&= \frac{\left\langle \frac{\partial \Upsilon}{\partial p}((p,v),x)\cdot w, \Upsilon((p,v),x) \right\rangle + \langle \Upsilon(p,\xi,x), \Upsilon((p,v),x)\rangle}{|\Upsilon((p,v),x)|}
\end{aligned}$$

for every $w, \xi \in \mathbf{R}^2$ (notice that Υ is linear with respect to v).

Therefore, if we fix $\tau > 0$ and take $0 < |v| \leqslant \tau$ together with $|w| \leqslant 1$ and $|\xi| \leqslant 1$, we get

$$\begin{aligned}
\left|\frac{\partial|\Upsilon|}{\partial(p,v)}((p,v),x)\cdot(w,\xi)\right| &\leqslant \left|\frac{\partial \Upsilon}{\partial p}((p,v),x)\cdot w\right| + |\Upsilon((p,\xi),x)| \\
&\qquad \text{(by Cauchy-Schwarz inequality)} \\
&\leqslant \tau \left\| \frac{\partial \Upsilon}{\partial p}((p,\cdot),x) \right\| + \|\Upsilon((p,\cdot),x)\|,
\end{aligned}$$

where $\|\cdot\|$ and $|||\cdot|||$ are respectively the operator norms on $\mathrm{L}(\mathbf{R}^2, \mathbf{R}^2)$ and $\mathrm{L}_2(\mathbf{R}^2 \times \mathbf{R}^2, \mathbf{R}^2)$ (bilinear maps from $\mathbf{R}^2 \times \mathbf{R}^2$ to $\mathbf{R}^2$).

Since Υ is C^∞, the functions $(x,p) \mapsto \|\Upsilon((p,\cdot),x)\|$ and $(x,p) \mapsto \left|\left|\left| \dfrac{\partial \Upsilon}{\partial p}((p,\cdot),x) \right|\right|\right|$ are continuous on $\partial\mathcal{D} \times \mathcal{D}$. So, given any $r \in (0,R)$, the compactness of $\partial\mathcal{D} \times \overline{\mathcal{D}(r)}$ implies that there exist positive constants Λ_1 and Λ_2 such that

$$\|\Upsilon((p,\cdot),x)\| \leqslant \Lambda_1 \quad \text{and} \quad \left|\left|\left| \frac{\partial \Upsilon}{\partial p}((p,\cdot),x) \right|\right|\right| \leqslant \Lambda_2$$

for all $(x,p) \in \partial\mathcal{D} \times \mathcal{D}(r)$.

Conclusion: for any $p \in \mathcal{D}(r)$, $v \in \mathbf{R}^2$ such that $0 < |v| \leqslant \tau$, $x \in \partial\mathcal{D} \smallsetminus \{e^-(p,v), e^+(p,v)\}$ and $w, \xi \in \mathbf{R}^2$ with $|w| \leqslant 1$ and $|\xi| \leqslant 1$, we have

$$\left|\frac{\partial|\Upsilon|}{\partial(p,v)}((p,v),x)\cdot(w,\xi)\right| \leqslant \tau\Lambda_1 + \Lambda_2.$$

Now, since $\{e^-(p,v), e^+(p,v)\}$ is a set of zero measure with respect to $\mathrm{d}x$, we obtain from Lebesgue's dominated convergence theorem (see for example [10], page 123) that the Finsler metric F on $\mathcal{D}$ is differentiable on $\mathcal{D}(r) \times \{v \in \mathbf{R}^2 \mid 0 < |v| < \tau\}$.

As this holds for arbitrary $r \in (0,R)$ and $\tau > 0$, we eventually get that F is differentiable on $\mathcal{D} \times (\mathbf{R}^2 \smallsetminus \{0\}) = T\mathcal{D} \smallsetminus \{0\}$ with

$$\frac{\partial \mathsf{F}}{\partial(p,v)}(p,v) = \int_{\partial\mathcal{D}} \frac{\partial|\Upsilon|}{\partial(p,v)}((p,v),x)\mathrm{d}x.$$

Finally, using the same reasonning as above, one can show by induction that for every $n \in \mathbf{N}$ the Finsler metric F is n times differentiable on $T\mathcal{D} \smallsetminus \{0\}$ with

$$\frac{\partial^n \mathsf{F}}{\partial(p,v)^n}(p,v) = \int_{\partial\mathcal{D}} \frac{\partial^n|\Upsilon|}{\partial(p,v)^n}((p,v),x)\mathrm{d}x.$$

This proves that F is smooth. $\square$

Now, using Theorem 3.1, our aim is to construct a 'good' family $(\mathfrak{S}_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$ of admissible sets of parameterized curves for $\mathcal{D}$ satisfying Property (C), from which we shall later be able to get a family $(\mathsf{F}_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$ of Finsler metrics on $\mathbf{H}^2$ as needed in the Main Theorem. But, as we already mentionned, the main difficulty will be to ensure these Finsler metrics be *smooth* on the *whole* space $\mathbf{H}^2$ (and not only on the disk $\mathcal{D}$) and *coincide* with the Riemannian hyperbolic metric g_0 *outside* $\mathcal{D}$. Given any $\lambda \in (-\varepsilon,\varepsilon)$ and according to Theorem 3.1, it seems reasonable to ask all the parameterized curves in $\mathfrak{S}_\lambda$ to coincide with the geodesics for g_0 in a neighborhood of $\partial\mathcal{D}$ (note that since $(\mathbf{H}^2, g_0)$ has been chosen to be the Klein model of the hyperbolic plane, the images of the g_0-geodesics are affine segments, thus very easy to be dealt with). However, we also want the Finsler metric F_λ *not* to be Riemannian, and this will be the case if we choose $\mathfrak{S}_\lambda$ to be the set of the parameterized curves obtained as a 'barycenter' of the geodesics for g_0 and the geodesics for some 'good' Riemannian metric g_λ conformal to g_0.

The advantage in constructing a family $(\mathfrak{S}_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$ in this way is that all the Finsler metrics of the associated family $(\mathsf{F}_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$ obtained by Arcostanzo's theorem will satisfy the desired properties listed in the Main Theorem, but this construction will have a cost. Indeed, proving that the set $\mathfrak{S}_\lambda$ of parameterized curves is *admissible* for $\mathcal{D}$ and *has* Property (C) is not easy and will be done at the expense of great effort. This is why we will have to make very technical considerations just in order to ensure admissibility and Property (C) for the family $(\mathfrak{S}_\lambda)_{\lambda\in(-\varepsilon,\varepsilon)}$.

3.2. **Constructing a family of admissible sets of parameterized curves.**

3.2.1. The setting.

We will now construct a family of admissible sets of parameterized curves for $\mathcal{D}$ by interpolating between the maximal geodesics for the hyperbolic metric g_0 on $\mathbf{H}^2$ and those for a nearby Riemannian metric of non-constant curvature that is conformal to g_0.

More precisely, let Δ be the Laplacian for g_0 and fix a regular eigenfunction $\psi : \overline{\mathcal{D}} \longrightarrow \mathbf{R}$ of Δ on $\overline{\mathcal{D}}$ associated with the first eigenvalue a of Δ and satisfying the Dirichlet condition $\psi_{|\partial\mathcal{D}} \equiv 0$. It is then well known that $a > 0$ and that ψ can be chosen to be positive on $\mathcal{D}$. Furthermore, as g_0 is invariant under the group $\mathrm{O}(\mathbf{R}^2)$ of linear Euclidean isometries (*i.e.*, $A^*g_0 = g_0$ for all $A \in \mathrm{O}(\mathbf{R}^2)$) thanks to Equation 3.1, we get that ψ is $\mathrm{O}(\mathbf{R}^2)$-invariant.

Next, let $\theta : \mathbf{H}^2 \longrightarrow \mathbf{R}$ be any C^∞ function that is $\mathrm{O}(\mathbf{R}^2)$-invariant and such that $\theta \equiv 1$ on $\overline{\mathcal{D}(R/4)}$ and $\theta \equiv 0$ on $\mathbf{H}^2 \smallsetminus \mathcal{D}(R/2)$. The new function $f : \mathbf{H}^2 \longrightarrow \mathbf{R}$ defined by

$$f(p) = \begin{cases} \psi(p)\theta(p) & \text{if } p \in \mathcal{D} \\ 0 & \text{if } p \in \mathbf{H}^2 \smallsetminus \mathcal{D} \end{cases}$$

is thus C^∞ and $\mathrm{O}(\mathbf{R}^2)$-invariant, together with $f \equiv 0$ on $\mathbf{H}^2 \smallsetminus \mathcal{D}(R/2)$ and $\Delta f = af$ on $\mathcal{D}(R/4)$. In particular, since $a > 0$ and ψ is positive on $\mathcal{D}$, there exists a number $\delta_0 > 0$ such that $(\Delta f)(p) \geqslant 1/\delta_0^2$ for all $p \in \mathcal{D}(R/4)$.

Proposition 3.1. *The function*

$$\begin{array}{rcl} \alpha : (-\delta_0, \delta_0) \times \mathbf{H}^2 & \longrightarrow & \mathbf{R} \\ (\lambda, p) & \longmapsto & \alpha(\lambda, p) = \alpha_\lambda(p) := e^{2\lambda^2 f(p)} \end{array}$$

is C^∞ *and satisfies the following:*

(1) $\alpha_0 \equiv 1$*;*

(2) for all $\lambda \in (-\delta_0, \delta_0)$ *and* $p \in \mathbf{H}^2 \smallsetminus \mathcal{D}(R/2)$*, we have* $\alpha_\lambda(p) = 1$*; and*

(3) for all $\lambda \in (-\delta_0, \delta_0)$ *with* $\lambda \neq 0$*, the Riemannian metric* $g_\lambda : \mathbf{H}^2 \longrightarrow \mathrm{Sym}_2(T\mathbf{H}^2)$ *defined by* $g_\lambda(p) = \alpha_\lambda(p) g_0(p)$ *is* C^∞*, complete, and has* non-constant negative *Gaussian curvature on* any *neighborhood about* 0 *in* $\mathbf{H}^2$.

Proof.

The only two things to be proved deal with completeness and Gaussian curvature, since all the other points are clear.

So, fix $\lambda \in (-\delta_0, \delta_0) \smallsetminus \{0\}$.

• Step 1: To prove g_λ is complete, we will use the Hopf-Rinow theorem.

Let X be a closed set in $\mathbf{H}^2$ that is bounded for g_λ, and prove it is compact.

We have $X = X_1 \cup X_2$ with $X_1 = X \cap \overline{\mathcal{D}}$ and $X_2 = X \cap (\mathbf{H}^2 \smallsetminus \mathcal{D})$. As X_1 is closed in the compact set $\overline{\mathcal{D}}$, it is compact.

On the other hand, X_2 is included in the open set $(\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)})$ of $\mathbf{H}^2$ on which g_λ coincide with g_0. So, X_2 is bounded for g_0, and hence compact since the Klein metric g_0 is complete.

Conclusion: $X = X_1 \cup X_2$ is compact.

• Step 2: The Gaussian curvature K_λ of the metric g_λ depends on that $K_0 \equiv -1$ of g_0 according to the formula $\alpha_\lambda K_\lambda = K_0 - \Delta(\ln(\alpha_\lambda))/2$ (see for example [12], page 97), which implies $K_\lambda(p) = -(1 + \lambda^2(\Delta f)(p))e^{-2\lambda^2 f(p)}$ for all $p \in \mathbf{H}^2$. Thus, for every $p \in \mathcal{D}(R/4)$, we have $K_\lambda(p) < 0$ since $1 + \lambda^2(\Delta f)(p) \geqslant 1 - \lambda^2/\delta_0^2 > 0$.

On the other hand, given $r \in (0, R/4)$, if K_λ were constant on $\overline{\mathcal{D}(r)}$, there would exist $C \in \mathbf{R}$ such that for all $p \in \overline{\mathcal{D}(r)}$,

$$\ln(1 + \lambda^2(\Delta f)(p)) = 2\lambda^2 f(p) + C,$$

and hence

$$\ln(1 + a\lambda^2 f(p)) = 2\lambda^2 f(p) + C \tag{3.2}$$

since $\Delta f = af$ on $\overline{\mathcal{D}(r)} \subseteq \mathcal{D}(R/4)$ by construction of f.

Defining $t_0 := \min\{f(p) \mid p \in \overline{\mathcal{D}(r)}\}$ and $t_1 := \max\{f(p) \mid p \in \overline{\mathcal{D}(r)}\}$, Equation 3.2 writes

$$\ln(1 + a\lambda^2 t) = 2\lambda^2 t + C \tag{3.3}$$

for all $t \in [t_0, t_1] = f(\overline{\mathcal{D}(r)})$.

Since the function f coincides with $\psi > 0$ on $\mathcal{D}(R/4)$, it never vanishes on $\overline{\mathcal{D}(r)}$, and hence it cannot be constant on $\overline{\mathcal{D}(r)}$ (indeed, if f were constant on $\overline{\mathcal{D}(r)}$, then we would have $f = (\Delta f)/a \equiv 0$ on $\overline{\mathcal{D}(r)} \subseteq \mathcal{D}(R/4)$). Therefore we have $t_0 < t_1$, which makes sense to differentiate Equation 3.3 with respect to t and get

$$\frac{a\lambda^2}{1 + a\lambda^2 t} = 2\lambda^2$$

for all $t \in [t_0, t_1]$.

But this is impossible since $a \neq 0$ and $\lambda \neq 0$.

Conclusion: the Gaussian curvature K_λ cannot be constant on $\overline{\mathcal{D}(r)}$. □

Let us now show how we use Proposition 3.1 to construct a family $(\mathfrak{S}_\lambda)_{\lambda \in (-\delta_0, \delta_0)}$ of sets of parameterized curves $\gamma : [0, 1] \longrightarrow \mathbf{R}^2$ we will prove later they are admissible for $\mathcal{D}$ and have Property (C).

For each $\lambda \in (-\delta_0, \delta_0)$ and $x \in \mathbf{H}^2$, denote by $\exp_x^\lambda : T_x\mathbf{H}^2 = \mathbf{R}^2 \longrightarrow \mathbf{H}^2$ the exponential map at x associated with g_λ, and let $\exp^\lambda : T\mathbf{H}^2 = \mathbf{H}^2 \times \mathbf{R}^2 \longrightarrow \mathbf{H}^2 \times \mathbf{H}^2$ be defined by $\exp^\lambda(x, v) = (x, \exp_x^\lambda(v))$. Since g_λ is negatively curved, it has no conjugate points and thus $\exp^\lambda$ is a C^∞ diffeomorphism. In particular, g_λ is uniquely geodesic.

We next fix a C^∞ function $\rho : \mathbf{R} \longrightarrow [0,1]$ such that

$$
\begin{aligned}
&(1)\ \ \rho \equiv 1 \text{ on } [1/2, 2/3];\\
(3.4)\qquad &(2)\ \ \rho \equiv 0 \text{ on } [3/4, +\infty);\ \text{and}\\
&(3)\ \ \rho(t) = \rho(1-t) \text{ for all } t \in \mathbf{R}.
\end{aligned}
$$

Given any $\lambda \in (-\delta_0, \delta_0)$ and $x \in \mathbf{H}^2$, let $G_x^\lambda : \mathbf{H}^2 \times \mathbf{R} \longrightarrow \mathbf{H}^2$ and $\varphi_x^\lambda : \mathbf{H}^2 \times \mathbf{R} \longrightarrow \mathbf{R}^2$ be defined by

$$G_x^\lambda(y,t) := \exp^\lambda(x\,,\ t(\exp_x^\lambda)^{-1}(y))$$

and

$$\varphi_x^\lambda(y,t) := (1-\rho(t))G_x^0(y,t) + \rho(t)G_x^\lambda(y,t).$$

Roughly speaking, we obtain the parameterized curve $\varphi_x^\lambda(y,\cdot) : \mathbf{R} \longrightarrow \mathbf{R}^2$ as the 'barycenter' in $\mathbf{R}^2$ with 'weights' $1-\rho$ and ρ of the unique maximal geodesics $G_x^0(y,\cdot)$ and $G_x^\lambda(y,\cdot)$ for g_0 and g_λ respectively passing through x at $t=0$ and y at $t=1$ (see Figure 4).

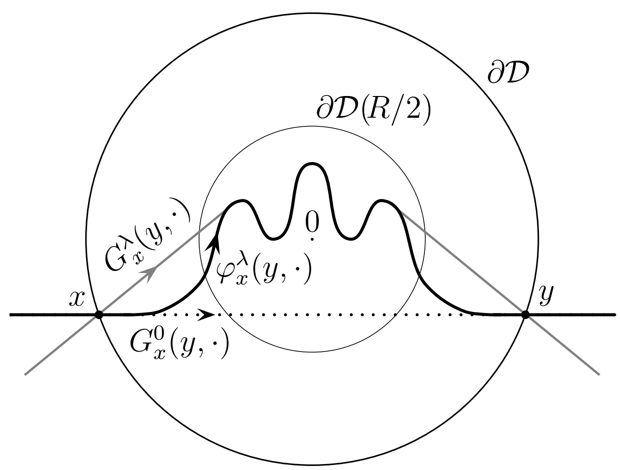

FIGURE 4. Constructing $\varphi_x^\lambda(y,\cdot)$

In the rest of this section, we prove that if we shrink $\delta_0 > 0$, then for each $\lambda \in (-\delta_0, \delta_0)$, the set of parameterized curves $\varphi_x^\lambda(y,\cdot) : [0,1] \longrightarrow \mathbf{R}^2$, where x and y are distinct points in $\partial\mathcal{D}$, is admissible for $\mathcal{D}$ and satisfies Property (C).

Then, in section 3.3, we prove these parameterized curves have additional properties that will be used to ensure that the Finsler metrics resulting from Theorem 3.1 satisfy our Main Theorem.

In the following technical lemma, we show that for any $\lambda \in (-\delta_0, \delta_0)$, if C is a closed convex set in $\mathbf{R}^2$ containing the open disk $\mathcal{D}(R/2)$, then $C \cap \mathbf{H}^2$ is in some sense convex with respect

to the set of parameterized curves $G_x^\lambda(y,\cdot) : [0,1] \longrightarrow \mathbf{H}^2$ (respectively $\varphi_x^\lambda(y,\cdot) : [0,1] \longrightarrow \mathbf{R}^2$), where $x, y \in C \cap \mathbf{H}^2$.

Lemma 3.1. *For each $\lambda \in (-\delta_0, \delta_0)$, we have*

(1) $\varphi_x^\lambda(y,t) = \varphi_y^\lambda(x, 1-t)$ for all $x, y \in \mathbf{H}^2$ and $t \in \mathbf{R}$,

(2) if C is any closed convex set in $\mathbf{R}^2$ such that $\mathcal{D}(R/2) \subseteq C$, then $G_x^\lambda(y,t) \in C$ and $\varphi_x^\lambda(y,t) \in C$ for all $x, y \in C \cap \mathbf{H}^2$ and $t \in [0,1]$,

(3) for all $x, y \in \partial\mathcal{D}$ and $t \in \mathbf{R}$, the equivalence $\varphi_x^\lambda(y,t) \in \mathcal{D} \iff t \in (0,1)$ holds.

Proof.

Fix $\lambda \in (-\delta_0, \delta_0)$.

• Point (1): Given any $x, y \in \mathbf{H}^2$, the parameterized curves $t \in \mathbf{R} \mapsto G_x^\lambda(y,t) \in \mathbf{H}^2$ and $t \in \mathbf{R} \mapsto G_y^\lambda(x, 1-t) \in \mathbf{H}^2$ are both g_λ-geodesics passing through x at $t = 0$ and y at $t = 1$. They are thus equal since g_λ is uniquely geodesic. Then point (1) follows from property (1) in Equation 3.4 satisfied by the function ρ.

• Point (2): Let C be a closed convex set in $\mathbf{R}^2$ such that $\mathcal{D}(R/2) \subseteq C$. Let $x, y \in C \cap \mathbf{H}^2$, and consider the g_λ-geodesic $\kappa : \mathbf{R} \longrightarrow \mathbf{H}^2$ defined by $\kappa(t) := G_x^\lambda(y,t)$.

We shall now prove by contradiction that the image of κ is included in C. Then, since the image of the g_0-geodesic $G_x^0(y,\cdot)$ is in C, we will get that the image of the interpolated curve $\varphi_x^\lambda(y,\cdot)$ is also in C. So, let us suppose that there exists $t_0 \in [0,1]$ such that $p_0 = \kappa(t_0) \notin C$ and prove this is not possible.

Let $\tau_0 := \max\{t \in [0,t_0] \mid \kappa(t) \in C\}$ and $\tau_1 := \min\{t \in [t_0, 1] \mid \kappa(t) \in C\}$ (note that τ_0 and τ_1 exist since $\kappa(0) = x \in C$ and $\kappa(1) = y \in C$).

Then $\tau_0 < t_0 < \tau_1$, and for all $t \in (\tau_0, \tau_1)$ we have $\kappa(t) \notin C$, which implies that $\kappa((\tau_0,\tau_1))$ is the affine segment $]\kappa(\tau_0), \kappa(\tau_1)[$ since the metric g_λ coincides with g_0 on the open set $\mathbf{H}^2 \smallsetminus C$ of $\mathbf{H}^2$ (recall images of the g_0-geodesics are affine segments). But $]\kappa(\tau_0), \kappa(\tau_1)[\subseteq [\kappa(\tau_0), \kappa(\tau_1)] \subseteq C$ since $\kappa(\tau_0), \kappa(\tau_1) \in C$ and C is convex. As $p_0 = \kappa(t_0) \in \kappa((\tau_0,\tau_1)) =]\kappa(\tau_0), \kappa(\tau_1)[$, we get a contradiction. So, $\kappa([0,1]) \subseteq C$.

On the other hand, the image of $[0,1]$ under the g_0-geodesic $G_x^0(y,\cdot)$ is the affine segment $[x,y]$, which lies in C since $x, y \in C$ and C is convex.

Finally, for all $t \in [0,1]$, the barycenter point $(1-\rho(t))G_x^0(y,t) + \rho(t)G_x^\lambda(y,t) = \varphi_x^\lambda(y,t)$ is contained in the convex set C.

• Point (3): Let $x, y \in \partial\mathcal{D}$ such that $x \neq y$ (the case $x = y$ is trivial).

To prove the $\Longrightarrow$ part, we show that $\varphi_x^\lambda(y,t) \notin \mathcal{D}$ for all $t \in \mathbf{R} \smallsetminus (0,1)$. The idea consists here in saying that if the parameterized curve $\varphi_x^\lambda(y,\cdot)$ leaves the disk $\mathcal{D}$, then it is equal to a g_0-geodesic. Hence, since the image of any geodesic for g_0 is an affine segment, $\varphi_x^\lambda(y,\cdot)$ can never go back into $\mathcal{D}$.

So, let $c : \mathbf{R} \longrightarrow \mathbf{H}^2$ be the g_0-geodesic defined by $c(t) := G_x^0(y,t)$. As the images of g_0-geodesics are affine segments, we can write $c(t) = x + \theta(t)(y-x)$ for all $t \in \mathbf{R}$, where

$\theta : \mathbf{R} \longrightarrow \mathbf{R}$ is a C^∞ function. Since c is a regular parameterized curve (it is a non-constant geodesic for a Riemannian metric) satisfying $c(0) = x$ and $c(1) = y$, the derivative of θ never vanishes and we have $\theta(0) = 0$ and $\theta(1) = 1$. Therefore θ is an increasing homeomorphism with $\theta([0,1]) = [0,1]$.

This implies that $c(\mathbf{R} \smallsetminus (0,1))$ is equal to the complement of the affine segment $]x,y[$ in the intersection of the straight line (xy) with $\mathbf{H}^2$. Since $]x,y[$ is the intersection of (xy) with $\mathcal{D}$, we get the inclusion $c(\mathbf{R} \smallsetminus (0,1)) \subseteq \mathbf{H}^2 \smallsetminus \mathcal{D}$.

But $\varphi_x^\lambda(y,t) = c(t)$ for all $t \in \mathbf{R} \smallsetminus (0,1)$ since $\rho \equiv 0$ on $\mathbf{R} \smallsetminus (0,1)$ by property (2) in Equation 3.4, and thus $\varphi_x^\lambda(y,\cdot)(\mathbf{R} \smallsetminus (0,1)) = c(\mathbf{R} \smallsetminus (0,1)) \subseteq \mathbf{H}^2 \smallsetminus \mathcal{D}$.

This establishes the $\Longrightarrow$ part in point (3).

To prove the $\Longleftarrow$ part, let $\nu : \mathbf{R} \longrightarrow \mathbf{H}^2$ be the g_λ-geodesic defined by $\nu(t) := G_x^\lambda(y,t)$.

Applying point (2) with $C = \overline{\mathcal{D}}$, we already have $\nu((0,1)) \in \overline{\mathcal{D}}$. Then suppose there exists $t_0 \in (0,1)$ such that $p_0 = \nu(t_0) \in \partial\mathcal{D}$ and prove this is not possible.

Since p_0 lies in the open set $\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)}$ of $\mathbf{H}^2$, the continuity of ν at t_0 implies there exists $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subseteq [0,1]$ and $\nu([t_0 - \varepsilon, t_0 + \varepsilon]) \subseteq \mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)}$. But g_λ agrees with g_0 on $\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)}$, so $\nu(t) = G_x^0(y,t)$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, and thus $\nu([t_0 - \varepsilon, t_0 + \varepsilon])$ is the affine segment $[\nu(t_0 - \varepsilon), \nu(t_0 + \varepsilon)]$. Hence $\nu(t_0 - \varepsilon), \nu(t_0 + \varepsilon) \in \overline{\mathcal{D}}$ with $p_0 \in [\nu(t_0 - \varepsilon), \nu(t_0 + \varepsilon)]$, which is impossible since $p_0 \in \partial\mathcal{D}$ is an extreme point for the convex set $\overline{\mathcal{D}}$. This shows that $\nu((0,1)) \subseteq \mathcal{D}$.

On the other hand, the image of $(0,1)$ under the g_0-geodesic $G_x^0(y,\cdot)$ is the affine segment $]x,y[$, which lies in $\mathcal{D}$ since $x, y \in \partial\mathcal{D}$ and $\mathcal{D}$ is strictly convex.

Finally, for all $t \in (0,1)$, the barycenter point $(1 - \rho(t))G_x^0(y,t) + \rho(t)G_x^\lambda(y,t) = \varphi_x^\lambda(y,t)$ is contained in the convex set $\mathcal{D}$. $\square$

We now consider the C^∞ map $\Phi : (-\delta_0, \delta_0) \times \partial\mathcal{D} \times \partial\mathcal{D} \times \mathbf{R} \longrightarrow \partial\mathcal{D} \times \mathbf{R}^2$ defined by

$$\Phi(\lambda, (x,y,t)) = \Phi_\lambda(x,y,t) := (x, \varphi_x^\lambda(y,t))$$

and denote by $\Delta := \{(x,x) \mid x \in \partial\mathcal{D}\}$ the diagonal of $\partial\mathcal{D} \times \partial\mathcal{D}$.

Using this map Φ, we shall prove that for all $\lambda \in (-\delta_0, \delta_0)$, the set

$$\mathfrak{S}_\lambda := \{\gamma_{(x,y)}^\lambda \mid (x,y) \in (\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta\}$$

of C^∞ parameterized curves $\gamma_{(x,y)}^\lambda : [0,1] \longrightarrow \mathbf{R}^2$ defined by $\gamma_{(x,y)}^\lambda(t) := \varphi_x^\lambda(y,t)$ is *admissible* for $\mathcal{D}$ and *satisfies* Property (C) provided $\delta_0 > 0$ is *sufficiently* small.

3.2.2. Diffeomorphism property for Φ_λ.

Let $\mathcal{T} := \{(x,p) \in \partial\mathcal{D} \times \mathbf{H}^2 \mid p - x \in T_x\partial\mathcal{D}\} = \{(x,p) \in \partial\mathcal{D} \times \mathbf{H}^2 \mid \langle x, p-x\rangle = 0\}$, $M := ((\partial\mathcal{D} \times \partial\mathcal{D}) \setminus \Delta) \times (\mathbf{R}\setminus\{0\})$ and $N := (\partial\mathcal{D} \times \mathbf{H}^2)\setminus\mathcal{T}$.

The aim of this section is to prove the following:

Proposition 3.2.

(1) For every $\lambda \in (-\delta_0, \delta_0)$, we have $\Phi_\lambda(M) \subseteq N$.

(2) There is $a \in (0, \delta_0)$ such that $\Phi_\lambda : M \longrightarrow N$ is a diffeomorphism for each $\lambda \in (-a, a)$.

Thanks to this *key* proposition and Corollary 3.1 below, we will be able to prove that the set $\mathfrak{S}_\lambda$ satisfies properties (1), (3) and (4) in Definition 3.1 (admissibility) together with point (2) in Definition 3.2 (Property (C)) after a suitable shrink of $a > 0$. Then, since property (2) in Definition 3.1 is obvious by construction of $\mathfrak{S}_\lambda$ and since point (1) in Definition 3.2 will be a consequence of Proposition 3.3 below, $\mathfrak{S}_\lambda$ will finally be a set of parameterized curves that is admissible for $\mathcal{D}$ and has Property (C).

Now, the argument to prove Proposition 3.2 consists in saying that since it is obviously true for $\lambda = 0$, it still remains true for any λ that is very close to 0.

In order to apply this perturbation argument in a rigorous way, we will make use of two classical results in algebraic and differential topology we recall here:

Lemma 3.2 (Covering maps. See [11], page 109)**.** *Let X and Y be Hausdorff topological spaces such that X is compact and Y is connected. Then any local homeomorphism $f : X \longrightarrow Y$ is a covering map with a finite number of sheets.*

and

Lemma 3.3 (Regular points. See [13], page 35)**.** *Let Λ, M and N be C^1 manifolds, and let*

$$\begin{array}{rcl} F : \Lambda \times M & \longrightarrow & N \\ (\lambda, x) & \longmapsto & F(\lambda, x) = f_\lambda(x) \end{array}$$

be a C^1 map. Let $\lambda_0 \in \Lambda$, $y_0 \in N$ and $K \subseteq M$ be a compact set. Then, if every $x \in f_{\lambda_0}^{-1}(y_0) \cap K$ is a regular point of f_{λ_0}, there exists an open neighborhood U of λ_0 in Λ such that for each $\lambda \in U$, any $x \in f_\lambda^{-1}(y_0) \cap K$ is a regular point of f_λ.

Proof of Proposition 3.2.

The proof will consist in five steps.

After showing that $\Phi_0(M) \subseteq N$ and $\Phi_0 : M \longrightarrow N$ is a bijection, we first prove that Φ_0 is a local diffeomorphism. We use this and Lemma 3.3 (Regular points Lemma) to find a value $a \in (0, \delta_0)$ such that for each $\lambda \in (-a, a)$, $\Phi_\lambda(M) \subseteq N$ and $\Phi_\lambda : M \longrightarrow N$ is also a local diffeomorphism. Next, we use Lemma 3.2 (Covering maps Lemma) to obtain that $\Phi_\lambda : M \longrightarrow N$ is a finite sheeted covering map. Finally we prove there is a point in N at

which the number of pre-images for this covering map is 1, and conclude $\Phi_\lambda : M \longrightarrow N$ is a diffeomorphism.

• Step 1: We begin by showing that $\Phi_0(M) \subseteq N$ and $\Phi_0 : M \longrightarrow N$ is a bijection.

Since the images of g_0-geodesics are affine segments, for each $(x,y,t) \in ((\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta) \times \mathbf{R}$, there is a unique real number $\omega(x,y,t)$ such that $\varphi_x^0(y,t) = G_x^0(y,t) = x + \omega(x,y,t)(y-x)$, and thus $\Phi_0(x,y,t) = (x\,,\, x + \omega(x,y,t)(y-x))$. The function $\omega : ((\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta) \times \mathbf{R} \longrightarrow \mathbf{R}$ is therefore C^∞ by smoothness of Φ_0, and satisfies the two following properties for each $(x,y) \in (\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta$:

(i) $\omega(x,y,0) = 0$ and $\omega(x,y,1) = 1$;

(ii) for any $t \in \mathbf{R}$, $\dfrac{\partial\omega}{\partial t}(x,y,t) \neq 0$.

For $x, y \in \partial\mathcal{D}$ with $x \neq y$ and $t \in \mathbf{R} \smallsetminus \{0\}$, we then have $\langle x\,,\, x + \omega(x,y,t)(y-x) - x\rangle = \omega(x,y,t)\,\langle x, y-x\rangle \neq 0$ since $\langle x, y-x\rangle \neq 0$ ($\partial\mathcal{D}$ is a Euclidean circle), $\omega(x,y,0) = 0$ (point (i) above) and $\omega(x,y,\cdot) : \mathbf{R} \longrightarrow \mathbf{R}$ is injective (point (ii) above). This shows $\Phi_0(M) \subseteq N$.

Now, given any $(x,p) \in N$, let y be the intersection point of the straight line (xp) with $\partial\mathcal{D}$. We have $y \neq x$, and thus we can write $p = x + \omega(x,y,t)(y-x)$ with a unique $t \in \mathbf{R}$ ($\omega(x,y,\cdot) : \mathbf{R} \longrightarrow \mathbf{R}$ is injective) which is not equal to 0 since $p \neq x$. This proves there is a unique $(x,y,t) \in M$ such that $\Phi_0(x,y,t) = (x,p)$. Hence $\Phi_0 : M \longrightarrow N$ is a bijection.

• Step 2: Let us prove $\Phi_0 : M \longrightarrow N$ is a local diffeomorphism.

Given any $(x,y,t) \in M$, it suffices to show that the linear tangent map $T_{(x,y,t)}\Phi_0 : T_{(x,y,t)}M \longrightarrow T_{\Phi_0(x,y,t)}N$ is injective since the manifolds M and N have the same dimension (equal to three).

But for all $(u,v,s) \in T_{(x,y,t)}M = T_x\partial\mathcal{D} \times T_y\partial\mathcal{D} \times \mathbf{R}$, we compute

$$T_{(x,y,t)}\Phi_0\cdot(u,v,s) = \left(u\,,\, u + \left\{\frac{\partial\omega}{\partial x}(x,y,t)\cdot u + \frac{\partial\omega}{\partial y}(x,y,t)\cdot v + s\frac{\partial\omega}{\partial t}(x,y,t)\right\}(y-x) + \omega(x,y,t)(v-u)\right).$$

So, if $T_{(x,y,t)}\Phi_0\cdot(u,v,s) = (0,0) \in T_{\Phi_0(x,y,t)}N = T_x\partial\mathcal{D} \times \mathbf{R}^2$, we get

$$u = 0 \quad \text{and} \quad u + \left\{\frac{\partial\omega}{\partial x}(x,y,t)\cdot u + \frac{\partial\omega}{\partial y}(x,y,t)\cdot v + s\frac{\partial\omega}{\partial t}(x,y,t)\right\}(y-x) + \omega(x,y,t)(v-u) = 0.$$

Hence $\left\{\dfrac{\partial\omega}{\partial y}(x,y,t)\cdot v + s\dfrac{\partial\omega}{\partial t}(x,y,t)\right\}(x-y) = \omega(x,y,t)v$.

As $v \in T_y\partial\mathcal{D}$, the first member of this equality lies in $T_y\partial\mathcal{D}$ too, which implies

$$\frac{\partial\omega}{\partial y}(x,y,t)\cdot v + s\frac{\partial\omega}{\partial t}(x,y,t) = 0 \tag{3.5}$$

since $x - y \notin T_y\partial\mathcal{D}$. Thus $\omega(x,y,t)v = 0$, that is $v = 0$ since $\omega(x,y,t) \neq 0$ (use $t \neq 0$ and points (i) and (ii) in Step 1).

Finally, replacing $v = 0$ in Equation 3.5, we obtain $s\dfrac{\partial \omega}{\partial t}(x, y, t) = 0$ and deduce $s = 0$ from point (ii) in Step 1.

Thus, $\Phi_0 : M \longrightarrow N$ is a local diffeomorphism.

• Step 3: Now we fix $\lambda \in (-\delta_0, \delta_0)$ and show by contradiction that $\Phi_\lambda(M) \subseteq N$.

Let $(x_0, p) \in \mathcal{T}$ and suppose there exist $y_0 \in \partial\mathcal{D} \smallsetminus \{x_0\}$ and $t_0 \in \mathbf{R} \smallsetminus \{0\}$ such that $\Phi_\lambda(x_0, y_0, t_0) = (x_0, p)$. Denoting by L the tangent line to $\partial\mathcal{D}$ at x_0, we have $p \in L$.

As $\rho \equiv 0$ on $(-\infty, 1/4] \cup [3/4, +\infty)$ by properties (2) and (3) in Equation 3.4, we have $\varphi^\lambda_{x_0}(y_0, t) = G^0_{x_0}(y_0, t)$ for all $t \in (-\infty, 1/4] \cup [3/4, +\infty)$. Thus, if we had $t_0 \in (-\infty, 1/4] \cup [3/4, +\infty)$, we would get $p = \varphi^\lambda_{x_0}(y_0, t_0) = G^0_{x_0}(y_0, t_0)$, and thus p would lie on the straight line (x_0y_0). But this implies $p = x_0$ since $(x_0y_0) \cap L = \{x_0\}$, and hence $x_0 = G^0_{x_0}(y_0, t_0)$, which means that $\Phi_0(x_0, y_0, t_0) = (x_0, x_0) = \Phi_0(x_0, y_0, 0)$.

Since $\Phi_0 : M \longrightarrow N$ is injective, we then get $t_0 = 0$, which is not possible. Therefore, we have $t_0 \in [1/4, 3/4]$. But this is also impossible since by point (3) of Lemma 3.1, $t_0 \in (0, 1)$ implies that $p = \varphi^\lambda_{x_0}(y_0, t_0) \in \mathcal{D}$, and $L \cap \mathcal{D}$ is empty.

• Step 4: Now, let $\ell := R/\sqrt{3} > 0$. Then any chord of $\partial\mathcal{D}$ that is tangent to $\partial\mathcal{D}(R/2)$ has a Euclidean length equal to 3ℓ.

Define $\Omega := \{(x, y) \in \partial\mathcal{D} \times \partial\mathcal{D} : |y - x| < \ell\}$ and consider the compact set $K := ((\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Omega) \times [1/4, 3/4] \subseteq M$.

The complement of K in M is the disjoint union of the open sets $U_1 := (\Omega \smallsetminus \Delta) \times (\mathbf{R} \smallsetminus \{0\})$ and $U_2 := ((\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Omega) \times ((-\infty, 0) \cup (0, 1/4) \cup (3/4, +\infty))$ of M.

We will first show that for each $\lambda \in (-\delta_0, \delta_0)$, $\Phi_\lambda : M \longrightarrow N$ induces a diffeomorphism from $M \smallsetminus K$ onto an open set in N. Then we will use Lemma 3.3 to find a number $a \in (0, \delta_0)$ such that for all $\lambda \in (-a, a)$, $\Phi_\lambda : M \longrightarrow N$ is a local diffeomorphism at any point in K.

Fix $\lambda \in (-\delta_0, \delta_0)$.

For every $(x, y) \in \Omega \smallsetminus \Delta$, the image of the g_0-geodesic $G^0_x(y, \cdot) : \mathbf{R} \longrightarrow \mathbf{H}^2$ lies in the open set $\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)}$ of $\mathbf{H}^2$ since this image is equal to the intersection of the straight line (xy) with $\mathbf{H}^2$. As the Riemannian metrics g_λ and g_0 coincide on $\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)}$, we get that the g_λ-geodesic $G^\lambda_x(y, \cdot) : \mathbf{R} \longrightarrow \mathbf{H}^2$ is actually equal to $G^0_x(y, \cdot) : \mathbf{R} \longrightarrow \mathbf{H}^2$. Hence, $(\Phi_\lambda)_{|U_1} = (\Phi_0)_{|U_1}$.

Next, using again the fact that $\rho \equiv 0$ on $(-\infty, 1/4] \cup [3/4, +\infty)$, we have $\varphi^\lambda_x(y, t) = G^0_x(y, t)$ for all $(x, y, t) \in \partial\mathcal{D} \times \partial\mathcal{D} \times ((-\infty, 1/4] \cup [3/4, +\infty))$. Thus, $(\Phi_\lambda)_{|U_2} = (\Phi_0)_{|U_2}$.

We conclude that we have $(\Phi_\lambda)_{|(M \setminus K)} = (\Phi_0)_{|(M \setminus K)}$, and hence $\Phi_\lambda : M \longrightarrow N$ induces a diffeomorphism from $M \smallsetminus K$ onto $\Phi_\lambda(M \smallsetminus K) = \Phi_0(M \smallsetminus K)$, which is an open set of N since $\Phi_0 : M \longrightarrow N$ is an open map by Step 2.

On the other hand, fixing $(x, p) \in N$, we have that the unique point $(x, y, t) \in M$ satisfying $\Phi_0(x, y, t) = (x, p)$ is regular for the diffeomorphism $\Phi_0 : M \longrightarrow N$, thus any point in $\Phi_0^{-1}((x, p)) \cap K$ is regular for Φ_0. We can then apply Lemma 3.3 to $\Phi : (-\delta_0, \delta_0) \times M \longrightarrow N$ and get the existence of a number $a \in (0, \delta_0)$ such that for each $\lambda \in (-a, a)$, all the points

in $\Phi_\lambda^{-1}((x,p)) \cap K$ are regular for $\Phi_\lambda : M \longrightarrow N$. As M and N have the same dimension, $\Phi_\lambda : M \longrightarrow N$ is a local diffeomorphism at any point in K.

Summing up, we proved that $\Phi_\lambda : M \longrightarrow N$ is a local diffeomorphism for every $\lambda \in (-a, a)$.

• Step 5: From now on, fix $\lambda \in (-a, a)$.

As $\Phi_\lambda : M \smallsetminus K \longrightarrow \Phi_\lambda(M \smallsetminus K)$ is a diffeomorphism, what remains is for us to show that the map $\Phi_\lambda : K \longrightarrow \Phi_\lambda(K)$ is one-to-one.

Since $\Phi_\lambda : K \longrightarrow \Phi_\lambda(K)$ is a local homeomorphism by Step 4 and K is compact and connected, we can apply Lemma 3.2 with $X = K$ and $Y = \Phi_\lambda(K)$ in order to get that $\Phi_\lambda : K \longrightarrow \Phi_\lambda(K)$ is a covering map with a finite number of sheets. We complete the argument by finding a point in the image of this covering map at which the number of pre-images is 1.

Choose $(x_0, y_0, t_0) \in K$ with $|y_0 - x_0| = 2\ell$ and let $p := \varphi_{x_0}^\lambda(y_0, t_0)$.

Since any chord of $\partial\mathcal{D}$ that is tangent to $\partial\mathcal{D}(R/2)$ has a Euclidean length equal to 3ℓ and since $|y_0 - x_0| < 3\ell$, the straight line $(x_0 y_0)$ does not intersect with $\overline{\mathcal{D}(R/2)}$. Then we have $\varphi_{x_0}^\lambda(y_0, t) = G_{x_0}^0(y_0, t)$ for all $t \in \mathbf{R}$, and thus $p = G_{x_0}^0(y_0, t_0)$. Consider any $y_1 \in \partial\mathcal{D} \smallsetminus \{x_0\}$ and $t_1 \in [1/4, 3/4]$ such that $p = \varphi_{x_0}^\lambda(y_1, t_1)$, and let us prove that $y_1 = y_0$ and $t_1 = t_0$.

Fix a closed half cone C in $\mathbf{R}^2$ whose vertex is x_0 and that contains $\mathcal{D}(R/2)$ with $y_0 \notin C$ (see Figure 5).

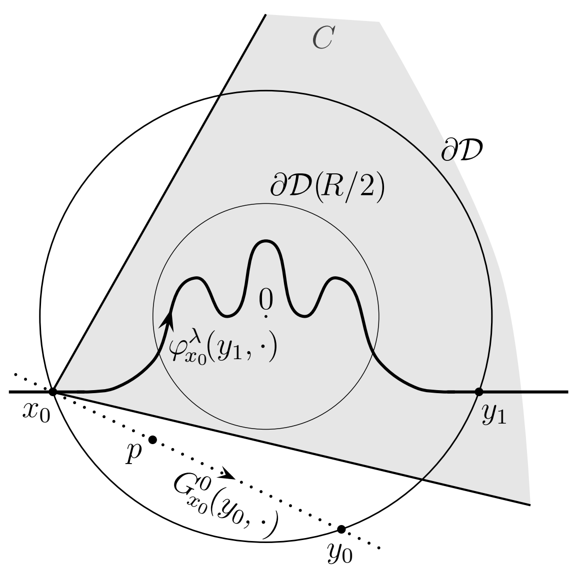

FIGURE 5. The map $\Phi_\lambda : K \longrightarrow \Phi_\lambda(K)$ is one-to-one

We show by contradiction that $y_1 \notin C$. If we assume y_1 is in C, then point (2) in Lemma 3.1 implies $p = \varphi_{x_0}^\lambda(y_1, t_1) \in C$ since $t_1 \in [0, 1]$. But this is not possible since $p \notin C$ (indeed, we have $y_0 \notin C$ and $p = G_{x_0}^0(y_0, t_0)$ lies on the affine segment $]x_0, y_0[$). We conclude that we necessarily have $y_1 \notin C$.

It follows that the straight line (x_0y_1) does not meet $\overline{\mathcal{D}(R/2)}$, and thus $\varphi^\lambda_{x_0}(y_1,t) = G^0_{x_0}(y_1,t)$ for all $t \in \mathbf{R}$. Therefore, $G^0_{x_0}(y_0,t_0) = p = \varphi^\lambda_{x_0}(y_1,t_1) = G^0_{x_0}(y_1,t_1)$, or equivalently $\Phi_0(x_0,y_0,t_0) = (x_0,p) = \Phi_0(x_0,y_1,t_1)$, which implies $y_1 = y_0$ and $t_1 = t_0$ since $\Phi_0 : M \longrightarrow N$ is injective.

In other words, we showed that $\Phi_\lambda^{-1}((x_0,p)) = \{(x_0,y_0,t_0)\}$ with $(x_0,y_0,t_0) \in K$. Hence $(x_0,p) \in \Phi_\lambda(K)$ and there is a unique point in the fiber of $\Phi_\lambda : K \longrightarrow \Phi_\lambda(K)$ over (x_0,p). This proves that the covering map $\Phi_\lambda : K \longrightarrow \Phi_\lambda(K)$ has only one sheet, which implies it is bijective.

But on the other hand, as we have seen that $(\Phi_\lambda)_{|(M \smallsetminus K)} = (\Phi_0)_{|(M \smallsetminus K)}$ and $\Phi_0 : M \longrightarrow N$ is a bijection, the map $\Phi_\lambda : M \smallsetminus K \longrightarrow \Phi_\lambda(M \smallsetminus K)$ is also a bijection.

Hence $\Phi_\lambda : M \longrightarrow N$ is bijective.

As we showed this map is a local diffeomorphism in Step 4, it is finally a diffeomorphism and this ends the proof of Proposition 3.2. ☐

For each $\lambda \in (-a,a)$, we can now define the map $\sigma_\lambda : N \longrightarrow \partial\mathcal{D}$ by $\sigma_\lambda(x,p) := y$, where $y \in \partial\mathcal{D}$ is such that (x,y,t) is the unique point in M that satisfies $\Phi_\lambda(x,y,t) = (x,p)$ according to Proposition 3.2 (see Figure 6).

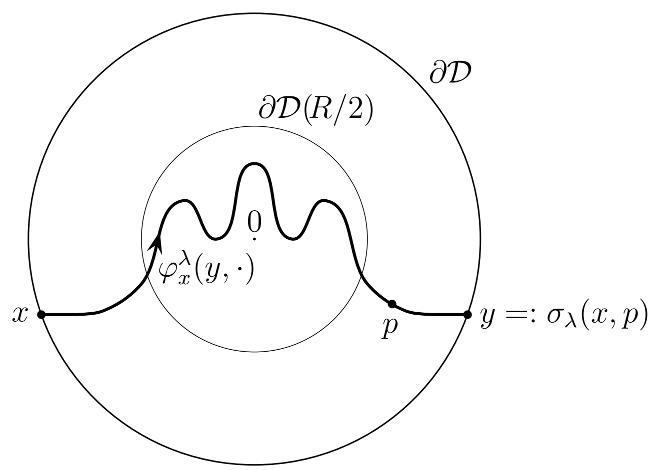

FIGURE 6. The 'end point map' σ_λ

Remark 3.4. By the first point in Lemma 3.1, for any $x, y, p \in \mathbf{H}^2$, we have the equivalence

$$((x,p) \in N \ \text{ and } \ \sigma_\lambda(x,p) = y) \iff ((y,p) \in N \ \text{ and } \ \sigma_\lambda(y,p) = x).$$

Let us now prove the following useful result:

Lemma 3.4. *Let Λ, M and N be C^k manifolds ($k \geqslant 1$ integer), and let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of C^k diffeomorphisms from M to N.*
If $\begin{array}{rcl} \theta : \Lambda \times M & \longrightarrow & N \\ (\lambda, x) & \longmapsto & f_\lambda(x) \end{array}$ *is of class C^k, then the map* $\begin{array}{rcl} h : \Lambda \times N & \longrightarrow & \Lambda \times M \\ (\lambda, x) & \longmapsto & (\lambda, f_\lambda^{-1}(x)) \end{array}$ *is a C^k diffeomorphism.*

In particular, $\begin{array}{rcl} \Lambda \times N & \longrightarrow & M \\ (\lambda, x) & \longmapsto & f_\lambda^{-1}(x) \end{array}$ *is of class C^k.*

Proof.

Since the map $\begin{array}{rcl} h : \Lambda \times M & \longrightarrow & \Lambda \times N \\ (\lambda, x) & \longmapsto & (\lambda, f_\lambda(x)) =: (\lambda, \theta(\lambda, x)) \end{array}$ is of class C^k and bijective, it suffices to show it is a local diffeomorphism. But this is equivalent to showing that for any $(\lambda, x) \in \Lambda \times M$, the linear tangent map $T_{(\lambda,x)}h : T_{(\lambda,x)}(\Lambda \times M) \longrightarrow T_{h(\lambda,x)}(\Lambda \times N)$ is injective since the manifolds M and N have the same dimension by hypothesis.

Now, for all $(\xi, v) \in T_{(\lambda,x)}(\Lambda \times M) = T_\lambda\Lambda \times T_x M$, we have

$$T_{(\lambda,x)}h\cdot(\xi, v) = (\xi\,,\, T_{(\lambda,x)}\theta\cdot(\xi, v)) = \left(\xi\,,\, \frac{\partial\theta}{\partial\lambda}(\lambda, x)\cdot\xi + \frac{\partial\theta}{\partial x}(\lambda, x)\cdot v\right) = \left(\xi\,,\, \frac{\partial\theta}{\partial\lambda}(\lambda, x)\cdot\xi + T_x f_\lambda\cdot v\right).$$

So, if $T_{(\lambda,x)}h\cdot(\xi, v) = (0, 0) \in T_{h(\lambda,x)}(\Lambda \times N) = T_\lambda\Lambda \times T_{f_\lambda(x)}N$, we get

$$\xi = 0 \quad \text{and} \quad \frac{\partial\theta}{\partial\lambda}(\lambda, x)\cdot\xi + T_x f_\lambda\cdot v = 0.$$

Hence $T_x f_\lambda\cdot v = 0$, which implies $v = 0$ since f_λ is a diffeomorphism.

Conclusion: h is a C^k diffeomorphism.

Therefore, if $\pi : \Lambda \times M \longrightarrow M$ denotes the natural projection, $\pi \circ h^{-1} : \Lambda \times N \longrightarrow M$ is of class C^k, which establishes Lemma 3.4. □

This lemma then implies

Proposition 3.3. *The map*

$$\begin{array}{rcl} (-a, a) \times N & \longrightarrow & \partial\mathcal{D} \\ (\lambda, (x, p)) & \longmapsto & \sigma_\lambda(x, p) \end{array} \quad \textit{is } \mathrm{C}^\infty.$$

Proof.

If we introduce the natural projection $\pi : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R} \longrightarrow \mathbf{R}^2$ onto the second factor, we can write $\sigma_\lambda(x, p) = \pi(\Phi_\lambda^{-1}(x, p))$ for all $(x, p) \in N$. Then, applying Lemma 3.4 with $\Lambda := (-a, a)$ and $f_\lambda := \Phi_\lambda$ (which is a diffeomorphism by point (2) in Proposition 3.2), we get the result. □

A direct consequence of Proposition 3.2 is the following:

Corollary 3.1. *Given any $\lambda \in (-a, a)$, we have*

(1) for every $(x, y) \in (\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta$, the C^∞ parameterized curve $\varphi_x^\lambda(y, \cdot) : \mathbf{R} \longrightarrow \mathbf{R}^2$ is regular and injective, and

(2) for every $(x,p) \in N$ and $V \in \mathbf{R}^2 \smallsetminus \{0\}$,

$$\frac{\partial \sigma_\lambda}{\partial p}(x,p)\cdot V = 0 \iff V \text{ and } \frac{\partial \varphi_x^\lambda}{\partial t}(y,t) \text{ are parallel vectors,}$$

where $p := \varphi_x^\lambda(y,t)$.

Proof.

• Point (1): Let $(x,y) \in (\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta$.

For any $t \in \mathbf{R}$, we have

$$\left(0\,,\, \frac{\partial \varphi_x^\lambda}{\partial t}(y,t)\right) = \frac{\partial \Phi_\lambda}{\partial t}(x,y,t) = T_{(x,y,t)}\Phi_\lambda\cdot(0,0,1) \neq (0,0)$$

since $T_{(x,y,t)}\Phi_\lambda : T_{(x,y,t)}M \longrightarrow T_{\Phi_\lambda(x,y,t)}N$ is one-to-one by Proposition 3.2, and therefore $\dfrac{\partial \varphi_x^\lambda}{\partial t}(y,t) \neq 0$. Hence $\varphi_x^\lambda(y,\cdot) : \mathbf{R} \longrightarrow \mathbf{R}^2$ is regular.

Let $t_0, t_1 \in \mathbf{R}$ such that $\varphi_x^\lambda(y,t_0) = \varphi_x^\lambda(y,t_1)$. Then $\Phi_\lambda(x,y,t_0) = \Phi_\lambda(x,y,t_1)$. If $t_0 \neq 0$ and $t_1 \neq 0$, we have $(x,y,t_0),(x,y,t_1) \in M$, and thus $t_0 = t_1$ since $\Phi_\lambda : M \longrightarrow N$ is injective by Proposition 3.2. If $t_0 \neq 0$ and $t_1 = 0$, we get $\varphi_x^\lambda(y,t_0) = \varphi_x^\lambda(y,0)$ which also writes $\varphi_y^\lambda(x,1-t_0) = \varphi_y^\lambda(x,1)$ by point (1) in Lemma 3.1. Since $x \neq y$, we have $1-t_0 \neq 0$, and thus $1-t_0 = 1$ in the same way as previously, *i.e.*, $t_0 = 0 = t_1$.

This shows that $\varphi_x^\lambda(y,\cdot) : \mathbf{R} \longrightarrow \mathbf{R}^2$ is injective.

• Point (2): Let $(x,p) \in N$ and $V \in \mathbf{R}^2 \smallsetminus \{0\}$.

By Proposition 3.2, there are unique elements $(x,y,t) \in M$, $v \in T_y\partial\mathcal{D}$ and $s \in \mathbf{R}$ such that $(x,p) = \Phi_\lambda(x,y,t)$ and $(0,V) = T_{(x,y,t)}\Phi_\lambda\cdot(0,v,s)$.

Then we have the equivalences

$$\begin{aligned}
\frac{\partial \sigma_\lambda}{\partial p}(x,p)\cdot V = 0 \quad &\iff \quad T_{(x,p)}\sigma_\lambda\cdot(0,V) = 0\\
&\iff \quad T_{(x,y,t)}(\sigma_\lambda \circ \Phi_\lambda)\cdot(0,v,s) = 0\\
&\iff \quad v = 0\\
&\qquad\quad \text{(since } \sigma_\lambda(\Phi_\lambda(x,y,t)) = y\text{)}\\
&\iff \quad (0,V) = T_{(x,y,t)}\Phi_\lambda\cdot(0,0,s) = s\frac{\partial \Phi_\lambda}{\partial t}(x,y,t) = \left(0\,,\, s\frac{\partial \varphi_x^\lambda}{\partial t}(y,t)\right)\\
&\iff \quad V = s\frac{\partial \varphi_x^\lambda}{\partial t}(y,t),
\end{aligned}$$

and we are done. □

3.2.3. Admissibility and Property (C) for the set $\mathfrak{S}_\lambda$.

The following two propositions allow us to shrink $a > 0$ so that for *each* $\lambda \in (-a, a)$, the set of parameterized curves $\mathfrak{S}_\lambda = \{\gamma^\lambda_{(x,y)} \mid (x, y) \in (\partial\mathcal{D} \times \partial\mathcal{D})\setminus\Delta\}$ be admissible for $\mathcal{D}$ and have Property (C).

Proposition 3.4. *There exists a number $b \in (0, a)$ such that for all $\lambda \in (-b, b)$ and $p, q \in \overline{\mathcal{D}}$ with $p \neq q$, there is a unique $(x, y, t_0, t_1) \in ((\partial\mathcal{D}\times\partial\mathcal{D})\setminus\Delta)\times[0, 1]\times[0, 1]$ such that $p = \varphi^\lambda_x(y, t_0)$, $q = \varphi^\lambda_x(y, t_1)$ and $t_0 < t_1$.*

This proposition will imply that $\mathfrak{S}_\lambda$ satisfies property (3) in Definition 3.1 (admissibility) for every $\lambda \in (-b, b)$.

Proposition 3.5. *There exists a number $c \in (0, a)$ such that for all $\lambda \in (-c, c)$, $p \in \mathcal{D}$ and $V \in \mathbf{R}^2\setminus\{0\}$, there is a unique $(x, y, t) \in ((\partial\mathcal{D} \times \partial\mathcal{D})\setminus\Delta) \times (0, 1)$ such that $p = \varphi^\lambda_x(y, t)$ and $\dfrac{\partial\varphi^\lambda_x}{\partial t}(y, t)$ is parallel to V with the same direction.*

This proposition will imply that $\mathfrak{S}_\lambda$ satisfies property (4) in Definition 3.1 (admissibility) for every $\lambda \in (-c, c)$.

In order to prove these two results, we need the following classical lemma from differential topology. This lemma allows us to show certain properties that are true for $\lambda = 0$ continues to hold for $\lambda \in (-a, a)$ close enough to 0.

Lemma 3.5 (Regular value. See [14], Theorem 2.7)**.** *Let Λ, M and N be C^1 manifolds, and let*

$$\begin{array}{rcl} F : \Lambda \times M & \longrightarrow & N \\ (\lambda, x) & \longmapsto & F(\lambda, x) = f_\lambda(x) \end{array}$$

be a C^1 map. Given $y_0 \in N$, we have

(1) if y_0 is a regular value of f_λ for all $\lambda \in \Lambda$, then y_0 is a regular value of F,

(2) if y_0 is a regular value of F, then $W = F^{-1}(y_0)$ is a C^1 submanifold of $\Lambda \times M$ with dimension equal to $\dim(\Lambda) + \dim(M) - \dim(N)$, and we have the equivalence

$$\forall\lambda \in \Lambda,\ (y_0 \text{ is a regular value of } f_\lambda) \iff (\lambda \text{ is a regular value of } \pi_{|W} : W \longrightarrow \Lambda),$$

where $\pi : \Lambda \times M \longrightarrow \Lambda$ is the natural projection.

Proof of Proposition 3.4.

Fix two distinct points $p, q \in \overline{\mathcal{D}}$.

- Case 1: Suppose $p \in \partial\mathcal{D}$ (the case $q \in \partial\mathcal{D}$ is similar).

Then for each $\lambda \in (-a, a)$, there exist a unique $y \in \partial\mathcal{D}$ with $y \neq p$ and a unique $t_1 \in \mathbf{R}\setminus\{0\}$ such that $\Phi_\lambda(p, y, t_1) = (p, q)$ by point (2) in Proposition 3.2 since $(p, q) \in N$. Hence we have $p = \varphi^\lambda_p(y, 0)$ and $q = \varphi^\lambda_p(y, t_1)$ with $(p, y, 0) \in ((\partial\mathcal{D}\times\partial\mathcal{D})\setminus\Delta)\times[0, 1]$ and $t_1 \in (0, 1]$ by point (2) in Lemma 3.1 since $q \in \overline{\mathcal{D}}$.

Now, let $(x', y', t'_0, t'_1) \in ((\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta) \times [0,1] \times [0,1]$ be such that $p = \varphi^\lambda_{x'}(y', t'_0)$ and $q = \varphi^\lambda_{x'}(y', t'_1)$ with $t'_0 < t'_1$. If we had $t'_0 > 0$, then we would have $t'_0 \in (0,1)$, and therefore $p \in \mathcal{D}$ by point (3) in Lemma 3.1. But this is not true. So $t'_0 = 0$, which implies $x' = \varphi^\lambda_{x'}(y', 0) = p$, and hence $\Phi_\lambda(p, y', t'_1) = (p, q)$.

But we already had $\Phi_\lambda(p, y, t_1) = (p, q)$, thus $y' = y$ and $t'_1 = t_1$ since $t_1, t'_1 \in \mathbf{R} \smallsetminus \{0\}$ and $\Phi_\lambda : M \longrightarrow N$ is injective by point (2) in Proposition 3.2.

• Case 2: Suppose that both p and q are in $\mathcal{D}$.

Consider the function $F : (-a, a) \times \partial\mathcal{D} \longrightarrow \mathbf{R}$ defined by $F(\lambda, x) := f_\lambda(x) = \det\big(V_p^\lambda(x), V_q^\lambda(x)\big)$, where $V_p^\lambda(x) := \sigma_\lambda(x, p) - x$ and $V_q^\lambda(x) := \sigma_\lambda(x, q) - x$ (see Figure 7). Thanks to Proposition 3.3, this function is C^∞.

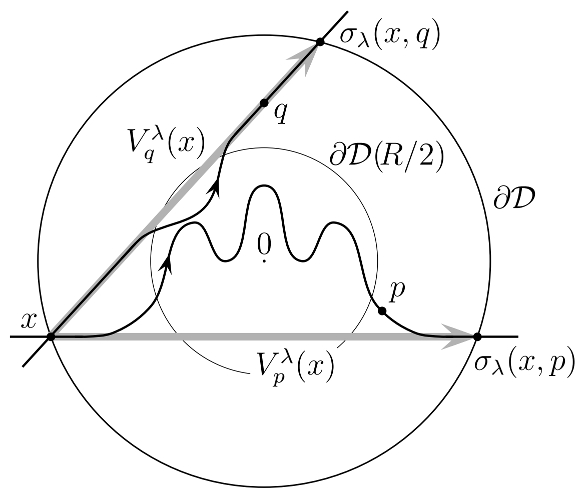

FIGURE 7. Proof of Proposition 3.4

Let x_0 and y_0 be the two intersection points of the strait line (pq) with $\partial\mathcal{D}$. As the images of g_0-geodesics are affine segments, we have $\sigma_0(x_0, p) = y_0 = \sigma_0(x_0, q)$, which shows that $V_p^0(x_0) = y_0 - x_0 = V_q^0(x_0)$ and thus $f_0(x_0) = 0$. Similarly $f_0(y_0) = 0$, and we actually have $f_0^{-1}(0) = \{x_0, y_0\}$. So, in order to prove 0 is a regular value of f_0, we have to show both x_0 and y_0 are regular points of f_0.

Let us prove x_0 is a regular point of f_0 (arguments are the same for y_0).

For any $u \in T_{x_0}\partial\mathcal{D}$, we have

$$\begin{aligned} T_{x_0} f_0 \cdot u &= \det\big(V_p^0(x_0)\,,\, T_{x_0} V_q^0 \cdot u\big) + \det\big(T_{x_0} V_p^0 \cdot u\,,\, V_q^0(x_0)\big) \\ &= \det\big(y_0 - x_0\,,\, T_{x_0} V_q^0 \cdot u - T_{x_0} V_p^0 \cdot u\big) \\ &= \det\left(y_0 - x_0\,,\, \frac{\partial \sigma_0}{\partial x}(x_0, q) \cdot u - \frac{\partial \sigma_0}{\partial x}(x_0, p) \cdot u\right). \end{aligned}$$

Now $\sigma_0(x,p) = p + \alpha(x)(p-x)$ and $\sigma_0(x,q) = q + \beta(x)(q-x)$ for any $x \in \partial\mathcal{D}$, where $\alpha : \partial\mathcal{D} \longrightarrow \mathbf{R}$ and $\beta : \partial\mathcal{D} \longrightarrow \mathbf{R}$ are functions that are C^∞ (since σ_0 is).

For any $u \in T_{x_0}\partial\mathcal{D}$ with $u \neq 0$, we have

$$\frac{\partial\sigma_0}{\partial x}(x_0,p)\cdot u - \frac{\partial\sigma_0}{\partial x}(x_0,p)\cdot u = (T_{x_0}\alpha\cdot u)(p-x_0) - (T_{x_0}\beta\cdot u)(q-x_0) + (\beta(x_0)-\alpha(x_0))u,$$

and thus

$$T_{x_0}f_0\cdot u = (\beta(x_0)-\alpha(x_0))\det((y_0-x_0\,,\,u)).$$

As u is not parallel to $y_0 - x_0$ by strict convexity of $\mathcal{D}$, the point x_0 will be regular for f_0 if $\beta(x_0) \neq \alpha(x_0)$. But if $\beta(x_0)$ were equal to $\alpha(x_0)$, we would get $y_0 = \sigma_0(x_0,p) = p+\alpha(x_0)(p-x_0)$ and $y_0 = \sigma_0(x_0,q) = q+\beta(x_0)(q-x_0)$. Therefore $(1-\alpha(x_0))(q-p) = 0$, and hence $\alpha(x_0) = \beta(x_0) = 1$ since $p \neq q$. This would then imply $2p = x_0 + y_0 = 2q$, contradicting the fact that p and q are distinct points. Thus 0 is a regular value of f_0.

Then, from Lemma 3.3 with $K := \partial\mathcal{D}$, there exists $b \in (0, a/2)$ such that 0 is a regular value of f_λ for all $\lambda \in (-2b, 2b)$. This implies by Lemma 3.5 that 0 is a regular value of $F_{|(-2b,2b)\times\partial\mathcal{D}}$, and hence that any $\lambda \in (-2b,2b)$ is a regular value of $\pi_{|W} : W \longrightarrow (-2b,2b)$, where $\pi : (-2b,2b)\times\partial\mathcal{D} \longrightarrow (-2b,2b)$ is the natural projection and $W = (F_{|(-2b,2b)\times\partial\mathcal{D}})^{-1}(0)$.

But $\dim(W) = 1 = \dim((-2b,2b))$, so $\pi_{|W}$ is a local diffeomorphism, thus a local homeomorphism, which implies that $\pi : W_0 \longrightarrow [-b,b]$ is also a local homeomorphism, where $W_0 := W \cap ([-b,b]\times\partial\mathcal{D})$.

Next, as W_0 is compact (since W is closed in $(-2b,2b)\times\partial\mathcal{D}$ and $[-b,b]\times\partial\mathcal{D}$ is compact) and $[-b,b]$ is connected, we get that $\pi : W_0 \longrightarrow [-b,b]$ is a covering map with a finite number of sheets by Lemma 3.2.

Since $\pi^{-1}(0)\cap W_0 = \{(0,x_0),(0,y_0)\}$, we have $\operatorname{card}(\pi^{-1}(\lambda)\cap W_0) = 2$ for all $\lambda \in [-b,b]$.

Hence, given $\lambda \in [-b,b]$, there are exactly two distinct points $x, y \in \partial\mathcal{D}$ such that

$$\det\big(V_p^\lambda(x), V_q^\lambda(x)\big) = 0 \quad\text{and}\quad \det\big(V_p^\lambda(y), V_q^\lambda(y)\big) = 0.$$

But this means that $\sigma_\lambda(x,p) = \sigma_\lambda(x,q) = y$ by Remark 3.4. So, there exist $t_0, t_1 \in \mathbf{R}$ such that $p = \varphi_x^\lambda(y,t_0)$ and $q = \varphi_x^\lambda(y,t_1)$ with $t_0 \leqslant t_1$ after a suitable labelling of x and y. As $\varphi_x^\lambda(y,\cdot) : \mathbf{R} \longrightarrow \mathbf{R}^2$ is injective by point (1) in Corollary 3.1, such t_0 and t_1 are unique with $0 < t_0 < t_1 < 1$.

This proves Proposition 3.4. □

Proof of Proposition 3.5.

Fix a point $p \in \mathcal{D}$ and a vector $V \in \mathbf{R}^2\smallsetminus\{0\}$, and consider the C^∞ function $F : (-a,a)\times\partial\mathcal{D} \longrightarrow \mathbf{R}$ defined by $F(\lambda,x) = f_\lambda(x) := \det\left(V\,,\,\dfrac{\partial\varphi_x^\lambda}{\partial t}(y,t)\right)$ with $\Phi_\lambda(x,y,t) = (x,p) \in N$. In other words, $f_\lambda(x) := \det\left(V\,,\,\dfrac{\partial\Phi_\lambda}{\partial t}(\Phi_\lambda^{-1}(x,p))\right)$ for all $(\lambda,x) \in (-a,a)\times\partial\mathcal{D}$ (see Figure 8).

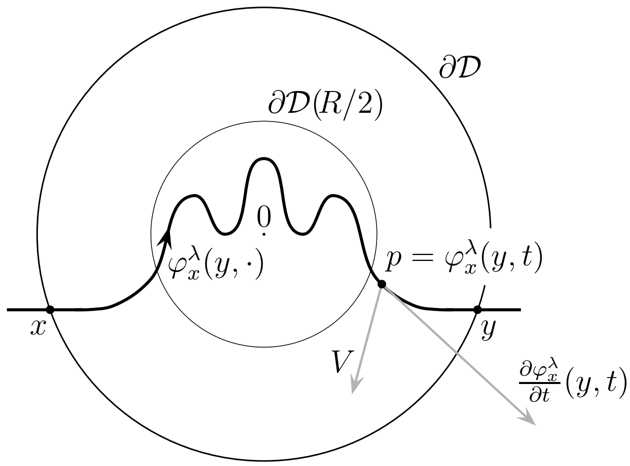

FIGURE 8. Proof of Proposition 3.5

Let x_0 and y_0 be the two intersection points of the straight line $p + \mathbf{R}V$ with $\partial\mathcal{D}$, and write $G_x^0(y,t) = x + \omega(x,y,t)(y-x)$ for all $(x,y,t) \in ((\partial\mathcal{D}\times\partial\mathcal{D})\smallsetminus\Delta)\times\mathbf{R}$, where ω is the function introduced in Step 1 in the proof of Proposition 3.2. As we have $f_0^{-1}(0) = \{x_0, y_0\}$, the value 0 will be regular for f_0 if x_0 and y_0 are regular points of f_0.

So, let us prove x_0 is a regular point of f_0 (arguments are the same for y_0).

Using the diffeomorphism $\Phi_0 : M \longrightarrow N$ (see point (2) in Proposition 3.2), we can write

$$f_0(x) = \det\left(V\,,\ \left\{\frac{\partial\omega}{\partial t}(\Phi_0^{-1}(x,p))\right\}(\sigma_0(x,p)-x)\right)$$

for all $x \in \partial\mathcal{D}$.

Thus, for any $u \in T_{x_0}\partial\mathcal{D}$, we have

$$\begin{aligned} T_{x_0}f_0\cdot u \;&=\; \det\left(V\,,\ \left\{T_{\Phi_0^{-1}(x_0,p)}\left(\frac{\partial\omega}{\partial t}\right)\cdot\{T_{(x_0,p)}\Phi_0^{-1}\cdot u\}\right\}(\sigma_0(x_0,p)-x_0)\right)\\ &\quad + \det\left(V\,,\ \frac{\partial\omega}{\partial t}(\Phi_0^{-1}(x_0,p))\left\{\frac{\partial\sigma_0}{\partial x}(x_0,p)\cdot u - u\right\}\right)\\ &=\; \det\left(V\,,\ \left\{T_{\Phi_0^{-1}(x_0,p)}\left(\frac{\partial\omega}{\partial t}\right)\cdot\{T_{(x_0,p)}\Phi_0^{-1}\cdot u\}\right\}(y_0-x_0)\right)\\ &\quad + \frac{\partial\omega}{\partial t}(\Phi_0^{-1}(x_0,p))\det\left(V\,,\ \frac{\partial\sigma_0}{\partial x}(x_0,p)\cdot u - u\right), \end{aligned}$$

that is,

$$T_{x_0}f_0\cdot u = \frac{\partial\omega}{\partial t}(\Phi_0^{-1}(x_0,p))\det\left(V\,,\ \frac{\partial\sigma_0}{\partial x}(x_0,p)\cdot u - u\right)$$

since $y_0 - x_0$ is parallel to V.

As $\dfrac{\partial \omega}{\partial t}(\Phi_0^{-1}(x_0,p)) \neq 0$ (from property (ii) for ω in Step 1 in the proof of Proposition 3.2), x_0 will be a regular point of f_0 if the vector $\dfrac{\partial \sigma_0}{\partial x}(x_0,p)\cdot u - u$ is not parallel to V for $u \in T_{x_0}\partial\mathcal{D}$ with $u \neq 0$. In order to prove this, just write $\sigma_0(x,p) = p + \alpha(x)(p-x)$ for any $x \in \partial\mathcal{D}$, where $\alpha : \partial\mathcal{D} \longrightarrow \mathbf{R}$ is a function that is C^∞ (since σ_0 is) and positive (since p is in the affine segment $]x, \sigma_0(x,p)[$). Then, for $u \in T_{x_0}\partial\mathcal{D}$ with $u \neq 0$, we have

$$\frac{\partial \sigma_0}{\partial x}(x_0,p)\cdot u - u = (T_{x_0}\alpha\cdot u)(p - x_0) - (1 + \alpha(x_0))u,$$

which is not parallel to V since $p - x_0$ is parallel to V and u is not (by strict convexity of $\mathcal{D}$).

Hence we have shown that 0 is a regular value of f_0, and we conclude exactly the same as in the end of the proof of Proposition 3.4 with c instead of b. $\square$

We can now use all we proved in this section to eventually obtain what we wanted:

Theorem 3.2. *There exists a number $\varepsilon_0 \in (0,a)$ such that for all $\lambda \in (-\varepsilon_0, \varepsilon_0)$, the set $\mathfrak{S}_\lambda := \{\gamma^\lambda_{(x,y)} \mid (x,y) \in (\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta\}$ of parameterized curves $\gamma^\lambda_{(x,y)} : [0,1] \longrightarrow \mathbf{R}^2$ defined by $\gamma^\lambda_{(x,y)}(t) := \varphi^\lambda_x(y,t)$ is admissible for $\mathcal{D}$ and satisfies Property (C).*

Proof.

Define $\varepsilon_0 := \dfrac{1}{2}\min\{b,c\} > 0$, where b and c are given respectively by Proposition 3.4 and Proposition 3.5, and let $\lambda \in (-\varepsilon_0, \varepsilon_0)$.

The fact that the set $\mathfrak{S}_\lambda$ is admissible for $\mathcal{D}$ follows from point (1) in Corollary 3.1, Proposition 3.4 and Proposition 3.5.

Property (C) for $\mathfrak{S}_\lambda$ is a consequence of Proposition 3.3 and point (2) in Corollary 3.1. $\square$

3.3. **Towards the Main Theorem.**

At this stage of the paper and following Arcostanzo's construction in [2], let us define for each $\lambda \in (-\varepsilon_0, \varepsilon_0)$ the function $\mathsf{F}_\lambda : T\mathcal{D} = \mathcal{D} \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ by setting

$$\mathsf{F}_\lambda(p,v) := \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(x, \sigma_\lambda(x,p))\left|\frac{\partial \sigma_\lambda}{\partial p}(x,p)\cdot v\right| \mathrm{d}x$$

for all $(p,v) \in T\mathcal{D}$.

Since the distance function d_{g_0} is C^∞ on $(\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta$ with $\frac{\partial^2 d_{g_0}}{\partial x \partial y}(x,y) > 0$ for all $(x,y) \in (\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta$ (point (2) in Remark 3.2), we get from Theorem 3.1 and Theorem 3.2 that F_λ is a *smooth* Finsler metric on $\mathcal{D}$ such that $d_{\mathsf{F}_\lambda} = d_{g_0}$.

On the other hand, since we have $\gamma^0_{(x,y)}(t) = \varphi^0_x(y,t) = G^0_x(y,t)$ for all $(x,y) \in (\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta$ and $t \in [0,1]$, the set $\mathfrak{G}_0$ coincides with the set of maximal geodesics of g_0 in $\overline{\mathcal{D}}$ after reparametrization by $[0,1]$. Thus, as mentionned in point (4) of Remark 3.2, F_0 *equals* the restriction to $T\mathcal{D}$ of the Finsler metric F_0 on $\mathbf{H}^2$ associated with g_0.

We are now going to give some properties about F_λ that will lead to the Main Theorem.

The first one shows that F_λ agrees with F_0 near the boundary $\partial\mathcal{D}$ of $\mathcal{D}$, which is not a surprise since our construction of F_λ has especially been made for this. Moreover, we prove that the region in $\mathcal{D}$ near the boundary of $\partial\mathcal{D}$ on which F_λ agrees with F_0 can actually be chosen in such a way that it does *not* depend on the parameter λ. This uniformity will later ensure that the family of Finsler metrics we will obtain in the Main Theorem is *smooth* with respect to λ.

Proposition 3.6. *There exists $R_0 \in (R/2, R)$ such that for every $\lambda \in (-\varepsilon_0, \varepsilon_0)$, the Finsler metric F_λ coincides with F_0 on $(\mathcal{D} \smallsetminus \overline{\mathcal{D}(R_0)}) \times \mathbf{R}^2$.*

In order to establish this fact, we will need the following useful lemma which proves that F_λ is invariant under the Euclidean isometries since all the objects we constructed so far have as much symmetry as the Euclidean circle $\partial\mathcal{D}$ has.

Lemma 3.6. *For any $\lambda \in (-\varepsilon_0, \varepsilon_0)$ and any linear Euclidean isometry $A \in \mathrm{O}(\mathbf{R}^2)$, we have $\mathsf{F}_\lambda(A(p), A(v)) = \mathsf{F}_\lambda(p, v)$ for all $p \in \mathcal{D}$ and $v \in \mathbf{R}^2$.*

Remark 3.5. Before proving this lemma, recall for the reader's convenience that the Klein metric g_0 whose associated Finsler metric is F_0 is given by

$$g_0(p)\cdot(v,v) = F_0(p,v)^2 = \frac{|v|^2}{1-|p|^2} + \frac{\langle v, p\rangle^2}{(1-|p|^2)^2}$$

for all $p \in \mathbf{H}^2$ and $v \in \mathbf{R}^2$.

Proof of Lemma 3.6.
Fix $\lambda \in \mathbf{R}$ and $A \in \mathrm{O}(\mathbf{R}^2)$.

For every $p \in \mathbf{H}^2$, we have $|A(p)| = |p|$, and thus $\alpha_\lambda(|A(p)|) = \alpha_\lambda(|p|)$. Since the Klein metric g_0 on $\mathbf{H}^2$ is invariant under A (*i.e.*, $A^*g_0 = g_0$) by the formula for g_0 in Remark 3.5, we get that g_λ is A-invariant too from the very definition of g_λ.

Hence, given any $x, y \in \mathbf{H}^2$ and any $t \in \mathbf{R}$, we have $G^0_{A(x)}(A(y),t) = A(G^0_x(y,t))$ and $G^\lambda_{A(x)}(A(y),t) = A(G^\lambda_x(y,t))$, which implies

$$\begin{aligned} \varphi^\lambda_{A(x)}(A(y),t) &= (1-\rho(t))G^0_{A(x)}(A(y),t) + \rho(t)G^\lambda_{A(x)}(A(y),t) \\ &= (1-\rho(t))A(G^0_x(y,t)) + \rho(t)A(G^\lambda_x(y,t)) \\ &= A\Big((1-\rho(t))G^0_x(y,t) + \rho(t)G^\lambda_x(y,t)\Big) \\ &= A(\varphi^\lambda_x(y,t)) \end{aligned} \tag{3.6}$$

for all $t \in \mathbf{R}$.

So, for every $\lambda \in (-\varepsilon_0, \varepsilon_0)$ and $(x,p) \in N$, we have $\sigma_\lambda(A(x), A(p)) = A(\sigma_\lambda(x,p))$, and therefore, for every $\lambda \in (-\varepsilon_0, \varepsilon_0)$, $p \in \mathcal{D}$ and $v \in \mathbf{R}^2$, one has

$$
\begin{aligned}
\mathsf{F}_\lambda(A(p), A(v)) &= \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(x, \sigma_\lambda(x, A(p)))\left|\frac{\partial \sigma_\lambda}{\partial p}(x, A(p))\cdot A(v)\right| \mathrm{d}x \\
&= \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(A(x), \sigma_\lambda(A(x), A(p)))\left|\frac{\partial \sigma_\lambda}{\partial p}(A(x), A(p))\cdot A(v)\right| \mathrm{d}x \\
&\quad \text{(since the canonical Euclidean measure } \mathrm{d}x \text{ on } \partial\mathcal{D} \text{ is } A\text{-invariant)} \\
&= \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(A(x), A(\sigma_\lambda(x, p)))\left|A(\frac{\partial \sigma_\lambda}{\partial p}(x, p)\cdot v)\right| \mathrm{d}x \\
&= \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(x, \sigma_\lambda(x, p))\left|\frac{\partial \sigma_\lambda}{\partial p}(x, p)\cdot v\right| \mathrm{d}x \\
&\quad \text{(since } d_{g_0} \text{ and } |\cdot| \text{ are } A\text{-invariant)} \\
&= \mathsf{F}_\lambda(p, v).
\end{aligned}
$$

This ends the proof of Lemma 3.6. $\square$

Proof of Proposition 3.6.

Applying Lemma 3.4 with $\Lambda := (-a, a)$ and $f_\lambda := \Phi_\lambda$ (which is a diffeomorphism by point (2) in Proposition 3.2), we get that $h : (-a,a) \times M \longrightarrow (-a,a) \times N$ defined by $h(\lambda, (x,y,t)) := (\lambda, (x, \varphi_x^\lambda(y,t)))$ is a diffeomorphism, hence a homeomorphism.

So, consider the open set $U := \{(x,y) \in \partial\mathcal{D} \times \partial\mathcal{D} \mid |x-y| > R\} \times (3/4, +\infty)$ in M, fix $x_0 \in \partial\mathcal{D}$, and define the compact set $K := \{y \in \partial\mathcal{D} \mid |x_0 - y| \geqslant \sqrt{3}R\} \subseteq \partial\mathcal{D}$.

Since $\{x_0\} \times K \times \{1\} \subseteq U$, the compact set

$$[-a/2, a/2] \times \{x_0\} \times K = h([-a/2, a/2] \times \{x_0\} \times K \times \{1\})$$

is included in the open set $\mathcal{U} := h((-a,a) \times U)$ of $(-a,a) \times N$. Thus, there exists a number $\tau_0 \in (0, R/2)$ such that $[-a/2, a/2] \times \{x_0\} \times \Sigma \subseteq \mathcal{U}$, where $\Sigma := \{(1+\tau)y \mid y \in K \text{ and } \tau \in (-\tau_0, \tau_0)\}$.

But Lemma 3.6 implies that for any $(\lambda, (x,p)) \in \mathcal{U}$ and $A \in \mathrm{O}(\mathbf{R}^2)$, we have $(\lambda, (A(x), A(p))) \in \mathcal{U}$. Hence, if

$$E := \{(x, (1+\tau)y) \mid x, y \in \partial\mathcal{D} \text{ and } |x-y| \geqslant \sqrt{3}R \text{ and } \tau \in (-\tau_0, \tau_0)\},$$

we get

$$[-a/2, a/2] \times E = \bigcup_{A \in \mathrm{O}(\mathbf{R}^2)} [-a/2, a/2] \times \{A(x_0)\} \times A(\Sigma) \subseteq \mathcal{U}. \tag{3.7}$$

Now define $R_0 := R - \tau_0 \in (R/2, R)$, and pick $\lambda \in [-a/2, a/2]$, $x \in \partial\mathcal{D}$ and $p \in \mathcal{D} \smallsetminus \overline{\mathcal{D}(R_0)}$.

Let $z \in \partial\mathcal{D}$ be the intersection point between $\partial\mathcal{D}$ and the open half line $x + \mathbf{R}_+^*(p-x)$.

If $|x-z| \geqslant \sqrt{3}R$, then $(x,p) \in E$, and thus $(\lambda,(x,p)) \in \mathcal{U}$ by Equation 3.7. This means that there are $y \in \partial\mathcal{D}$ and $t \in (3/4,+\infty)$ satisfying $p = \varphi_x^\lambda(y,t)$, which implies that $\sigma_\lambda(x,p) = y$. But, since $\rho \equiv 0$ on $(3/4,+\infty)$ by property (2) in Equation 3.4, we also have $p = \varphi_x^\lambda(y,t) = G_x^0(y,t) = \varphi_x^0(y,t)$, and hence $\sigma_0(x,p) = y$.

If $|x-z| < \sqrt{3}R$, then the image of the g_0-geodesic $G_x^0(z,\cdot) : \mathbf{R} \longrightarrow \mathbf{H}^2$ lies in the open set $\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)}$ of $\mathbf{H}^2$, since this image is equal to the intersection of the straight line (xz) with $\mathbf{H}^2$ and since any chord of $\partial\mathcal{D}$ that is tangent to $\partial\mathcal{D}(R/2)$ has a Euclidean length equal to $\sqrt{3}R$. Since the Riemannian metrics g_λ and g_0 coincide on $\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R/2)}$, we get that the g_λ-geodesic $G_x^\lambda(z,\cdot) : \mathbf{R} \longrightarrow \mathbf{H}^2$ is actually equal to the g_0-geodesic $G_x^0(z,\cdot) : \mathbf{R} \longrightarrow \mathbf{H}^2$. Thus, $\varphi_x^\lambda(z,t) = G_x^0(z,t) = \varphi_x^0(y,t)$ for all $t \in \mathbf{R}$. But the definition of z says that $p \in (xz)$, which means there is $t_0 \in \mathbf{R}$ such that $p = G_x^0(z,t_0)$. So, $p = \varphi_x^\lambda(z,t_0) = \varphi_x^0(z,t_0)$, and therefore $\sigma_\lambda(x,p) = z = \sigma_0(x,p)$.

Conclusion: for every $\lambda \in [-a/2,a/2]$, $x \in \partial\mathcal{D}$ and $p \in \mathcal{D} \smallsetminus \overline{\mathcal{D}(R_0)}$, we have $\sigma_\lambda(x,p) = \sigma_0(x,p)$.

Hence, for any $\lambda \in (-\varepsilon_0,\varepsilon_0) \subseteq [-a/2,a/2]$, $p \in \mathcal{D} \smallsetminus \overline{\mathcal{D}(R_0)}$ and $v \in \mathbf{R}^2$, we can write

$$\begin{aligned}
\mathsf{F}_\lambda(p,v) &= \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(x,\sigma_\lambda(x,p))\left|\frac{\partial\sigma_\lambda}{\partial p}(x,p)\cdot v\right| \mathrm{d}x \\
&= \frac{1}{4}\int_{\partial\mathcal{D}} \frac{\partial^2 d_{g_0}}{\partial x \partial y}(x,\sigma_0(x,p))\left|\frac{\partial\sigma_0}{\partial p}(x,p)\cdot v\right| \mathrm{d}x \\
&= \mathsf{F}_0(p,v) = F_0(p,v).
\end{aligned}$$

This proves Proposition 3.6. $\square$

From now on, for each $\lambda \in (-\varepsilon_0,\varepsilon_0)$ and thanks to Proposition 3.6, we extend the Finsler metric F_λ on the whole $\mathbf{H}^2$ by setting $\mathsf{F}_\lambda(p,v) = F_0(p,v)$ for all $(p,v) \in (\mathbf{H}^2 \smallsetminus \overline{\mathcal{D}(R_0)}) \times \mathbf{R}^2$.

Then we have

Proposition 3.7. *The family of Finsler metrics $(\mathsf{F}_\lambda)_{\lambda\in(-\varepsilon_0,\varepsilon_0)}$ on $\mathbf{H}^2$ satisfies the following:*

(1) the function $\Phi : (-\varepsilon_0,\varepsilon_0) \times T\mathbf{H}^2 \longrightarrow \mathbf{R}$ defined by $\Phi(\lambda,\cdot) := \mathsf{F}_\lambda(\cdot)$ for all $\lambda \in (-\varepsilon_0,\varepsilon_0)$ is continuous and C^∞ on $(-\varepsilon_0,\varepsilon_0) \times \mathbf{H}^2 \times (\mathbf{R}^2 \smallsetminus \{0\})$; and

(2) there exists $\varepsilon \in (0,\varepsilon_0)$ such that for each $\lambda \in (-\varepsilon,\varepsilon)$, the smooth Finsler metric F_λ is strongly convex and has no conjugate points.

Proof.

• Point (1): Consider the map $\Upsilon : (-\varepsilon_0,\varepsilon_0) \times \mathbf{H}^2 \times \mathbf{R}^2 \times \partial\mathcal{D} \longrightarrow \mathbf{R}^2$ defined by

$$\Upsilon((\lambda,p,v),x) := \frac{1}{4}\frac{\partial^2 d_{g_0}}{\partial x \partial y}(x,\sigma_\lambda(x,p))\frac{\partial\sigma_\lambda}{\partial p}(x,p)\cdot v.$$

Since d_{g_0} is C^∞ on $(\partial\mathcal{D}\times\partial\mathcal{D})\smallsetminus\{(x,x)\mid x\in\partial\mathcal{D}\}$ and $(\lambda,x,p)\mapsto\sigma_\lambda(x,p)$ is a C^∞ map from $(-\varepsilon_0,\varepsilon_0)\times\partial\mathcal{D}\times\mathcal{D}$ to $\partial\mathcal{D}$ by Proposition 3.3 which satisfies $\sigma_\lambda(x,p)\neq x$ for all $\lambda\in(-\varepsilon_0,\varepsilon_0)$ and $(x,p)\in\partial\mathcal{D}\times\mathcal{D}$, the positive function $(\lambda,x,p)\mapsto\dfrac{\partial^2 d_{g_0}}{\partial x\partial y}(x,\sigma_\lambda(x,p))$ is C^∞ on $(-\varepsilon_0,\varepsilon_0)\times\partial\mathcal{D}\times\mathcal{D}$, and therefore Υ is C^∞ on $(-\varepsilon_0,\varepsilon_0)\times\mathcal{D}\times\mathbf{R}^2\times\partial\mathcal{D}$.

Then, using the same arguments as in Remark 3.3, we get that the function Φ is continuous on $(-\varepsilon_0,\varepsilon_0)\times\mathcal{D}\times\mathbf{R}^2$ and C^∞ on $(-\varepsilon_0,\varepsilon_0)\times\mathcal{D}\times(\mathbf{R}^2\smallsetminus\{0\})$ since we have

$$\Phi(\lambda,(p,v))=\int_{\partial\mathcal{D}}|\Upsilon((\lambda,p,v),x)|\mathrm{d}x$$

for all $\lambda\in(-\varepsilon_0,\varepsilon_0)$ and $(p,v)\in T\mathbf{H}^2=\mathbf{H}^2\times\mathbf{R}^2$.

On the other hand, since $\Phi(\lambda,(p,v))=F_0(p,v)$ for all $\lambda\in(-\varepsilon_0,\varepsilon_0)$ and $(p,v)\in(\mathbf{H}^2\smallsetminus\overline{\mathcal{D}(R_0)})\times\mathbf{R}^2$ by construction, the function Φ is continuous on $(-\varepsilon_0,\varepsilon_0)\times(\mathbf{H}^2\smallsetminus\overline{\mathcal{D}(R_0)})\times\mathbf{R}^2$ and C^∞ on $(-\varepsilon_0,\varepsilon_0)\times(\mathbf{H}^2\smallsetminus\overline{\mathcal{D}(R_0)})\times(\mathbf{R}^2\smallsetminus\{0\})$.

Conclusion: Φ is a continuous function that is C^∞ on $(-\varepsilon_0,\varepsilon_0)\times\mathbf{H}^2\times(\mathbf{R}^2\smallsetminus\{0\})$.

• Point (2): As a consequence of the first point, the map $\lambda\mapsto\Phi(\lambda,\cdot)=\mathsf{F}_\lambda(\cdot)$ from $(-\varepsilon_0,\varepsilon_0)$ to $C^2(T\mathbf{H}^2\smallsetminus\{0\},\mathbf{R})$ is continuous when $C^2(T\mathbf{H}^2\smallsetminus\{0\},\mathbf{R})$ is endowed with the C^2-topology.

This first implies that $\frac{\partial^2\mathsf{F}_\lambda^2}{\partial v^2}$ is close to $\frac{\partial^2 F_0^2}{\partial v^2}$ in $C^0(T\mathbf{H}^2\smallsetminus\{0\},\mathbf{R})$ with respect to the C^0-topology whenever $\lambda\in(-\varepsilon_0,\varepsilon_0)$ is sufficiently small. Hence, there exists $\varepsilon_1\in(0,\varepsilon_0)$ such that F_λ is strongly convex for all $\lambda\in(-\varepsilon_1,\varepsilon_1)$ since the hyperbolic Finsler metric F_0 is.

Furthermore, if $\mathcal{V}\subseteq T(T\mathbf{H}^2)$ is the vertical vector bundle over $T\mathbf{H}^2$ (the kernel of the differential of the natural projection $T\mathbf{H}^2\longrightarrow\mathbf{H}^2$) and $\varphi_\lambda=(\varphi_\lambda^t)_{t\in\mathbf{R}}$ is the geodesic flow of F_λ on $T\mathbf{H}^2\smallsetminus\{0\}$ for any $\lambda\in(-\varepsilon_1,\varepsilon_1)$ (*i.e.*, the Euler-Lagrange flow of the non-degenerate Lagrangian $\mathcal{L}_\lambda:=\frac{1}{2}\mathsf{F}_\lambda^2:T\mathbf{H}^2\smallsetminus\{0\}\longrightarrow\mathbf{R}$), the map $\lambda\mapsto\varphi_\lambda$ from $(-\varepsilon_1,\varepsilon_1)$ to $C^1(\mathbf{R}\times(T\mathbf{H}^2\smallsetminus\{0\}),T\mathbf{H}^2\smallsetminus\{0\})$ is continuous when $C^1(\mathbf{R}\times(T\mathbf{H}^2\smallsetminus\{0\}),T\mathbf{H}^2\smallsetminus\{0\})$ is endowed with the C^1-topology.

Since the hyperbolic Finsler metric F_0 has no conjugate points, we have

$$\mathcal{V}_{(p,v)}\cap T_{\varphi_0^t(p,v)}\varphi_0^{-t}(\mathcal{V}_{\varphi_0^t(p,v)})\neq\{0\}\quad\text{for all}\quad(t,(p,v))\in\mathbf{R}\times(T\mathbf{H}^2\smallsetminus\{0\}).$$

Thus, there exists $\varepsilon\in(0,\varepsilon_1)$ such that

$$\mathcal{V}_{(p,v)}\cap T_{\varphi_\lambda^t(p,v)}\varphi_\lambda^{-t}(\mathcal{V}_{\varphi_\lambda^t(p,v)})\neq\{0\}\quad\text{for all}\quad(t,(p,v))\in\mathbf{R}\times(T\mathbf{H}^2\smallsetminus\{0\})\quad\text{and all}\quad\lambda\in(-\varepsilon,\varepsilon).$$

But this is equivalent to saying that the Finsler metric F_λ has no conjugate points whenever $\lambda\in(-\varepsilon,\varepsilon)$. ☐

Proposition 3.8. *For any $\lambda\in(-\varepsilon,\varepsilon)$, the Finsler metric F_λ is* not Riemannian *whenever $\lambda\neq 0$.*

Before proving this result, we will need to establish the following:

Lemma 3.7. *There exists $r_0\in(0,R/2)$ such that for every $\lambda\in(-\varepsilon,\varepsilon)$, all the geodesics of the restriction of the Riemannian metric g_λ to $\mathcal{D}(r_0)$ are geodesics for F_λ.*

Proof.

It will consists in four technical steps.

We first show that for any $\lambda \in \mathbf{R}$ and $x \in \partial\mathcal{D}$, the parameterized curve $\varphi_x^\lambda(-x, \cdot) : \mathbf{R} \longrightarrow \mathbf{R}^2$ passes through the origin 0 at $t = 1/2$ (here $y = -x \in \partial\mathcal{D}$ is the symmetric of x about 0). Then, remembering that $\varphi_x^\lambda(y, \cdot) = (1-\rho)G_x^0(y, \cdot) + \rho G_x^\lambda(y, \cdot)$ for all $y \in \partial\mathcal{D}$ and using the fact that $\rho \equiv 1$ on $[1/3, 2/3]$, we deduce the lemma.

• $\mathsf{Step\ 1}$: Fix arbitrary $\lambda \in \mathbf{R}$ and $x \in \mathbf{H}^2 \smallsetminus \{0\}$.

If $A \in \mathrm{O}(\mathbf{R}^2)$ is the Euclidean reflection through the line $x^\perp$, we have

$$\varphi_x^\lambda(-x, 1/2) = \varphi_{A(-x)}^\lambda(A(x), 1/2) = A(\varphi_{-x}^\lambda(x, 1/2))$$

on the one hand by Equation 3.6, and $\varphi_{-x}^\lambda(x, 1/2) = \varphi_x^\lambda(-x, 1/2)$ on the other hand by the first point in Lemma 3.1. Therefore $\varphi_x^\lambda(-x, 1/2) \in x^\perp$.

Next, if $B \in \mathrm{O}(\mathbf{R}^2)$ is the reflection through the line $\mathbf{R}x$, we have

$$\varphi_x^\lambda(-x, 1/2) = \varphi_{B(x)}^\lambda(B(-x), 1/2) = B(\varphi_x^\lambda(-x, 1/2))$$

by Equation 3.6, and hence $\varphi_{-x}^\lambda(x, 1/2) \in \mathbf{R}x$.

This shows that $\varphi_x^\lambda(-x, 1/2) = 0$.

• $\mathsf{Step\ 2}$: Now, as in the proof of Proposition 3.6, we will make use of the map $h : (-a, a) \times M \longrightarrow (-a, a) \times N$ defined by $h(\lambda, (x, y, t)) = (\lambda, (x, \varphi_x^\lambda(y, t)))$.

Consider the open set $V := ((\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta) \times (1/3, 2/3)$ in M, and define the compact set $L := \{(x, -x) \mid x \in \partial\mathcal{D}\} \subseteq \partial\mathcal{D} \times \partial\mathcal{D}$.

Since $L \times \{1/2\} \subseteq V$, the set $h([-a/2, a/2] \times L \times \{1/2\})$ is included in the open set $\mathcal{V} := h((-a, a) \times V)$ of $(-a, a) \times N$. But Step 1 implies that we have $[-a/2, a/2] \times \partial\mathcal{D} \times \{0\} \subseteq h([-a/2, a/2] \times L \times \{1/2\})$. So, using the compactness of $[-a/2, a/2] \times \partial\mathcal{D} \times \{0\}$, there exists a number $r_0 \in (0, R/2)$ such that $[-a/2, a/2] \times \partial\mathcal{D} \times \mathcal{D}(r_0) \subseteq \mathcal{V}$, which means that for every $\lambda \in [-a/2, a/2]$, $x \in \partial\mathcal{D}$ and $p \in \mathcal{D}(r_0)$, there are $z \in \partial\mathcal{D}$ and $\tau \in (1/3, 2/3)$ satisfying $p = \varphi_x^\lambda(z, \tau)$.

• $\mathsf{Step\ 3}$: For any $\lambda \in (-\varepsilon, \varepsilon) \subseteq [-a/2, a/2]$ and $x, y \in \partial\mathcal{D}$ with $x \neq y$, we have $\{t \in (0, 1) \mid \varphi_x^\lambda(y, t) \in \mathcal{D}(r_0)\} \subseteq (1/3, 2/3)$.

Indeed, if $t \in (0, 1)$ satisfies $p := \varphi_x^\lambda(y, t) \in \mathcal{D}(r_0)$, then by Step 2 there exist $z \in \partial\mathcal{D}$ and $\tau \in (1/3, 2/3)$ such that $\varphi_x^\lambda(y, t) = \varphi_x^\lambda(z, \tau)$. So $\Phi_\lambda(x, y, t) = \Phi_\lambda(x, z, \tau)$, and therefore $t = \tau \in (1/3, 2/3)$ since $\Phi_\lambda : M \longrightarrow N$ is injective by point (2) in Proposition 3.2.

• $\mathsf{Step\ 4}$: For this last step, fix $\lambda \in (-\varepsilon, \varepsilon)$, let $c : I \longrightarrow \mathbf{H}^2$ be a g_λ-geodesic such that $c(I) \subseteq \mathcal{D}(r_0)$, where $I \subseteq \mathbf{R}$ is an interval, and prove that c is also a F_λ-geodesic.

For doing this, choose arbitrary $s_0, s_1 \in I$ with $s_0 < s_1$, and define $p_0 := c(s_0)$ and $p_1 := c(s_1)$. By Proposition 3.4, there exists $(x, y, t_0, t_1) \in ((\partial\mathcal{D} \times \partial\mathcal{D}) \smallsetminus \Delta) \times (0, 1) \times (0, 1)$ such that $p_0 = \varphi_x^\lambda(y, t_0)$ and $p_1 = \varphi_x^\lambda(y, t_1)$ with $t_0 \leqslant t_1$.

Then, by Theorem 3.1, the parameterized curve $\kappa : [t_0, t_1] \subseteq (0,1) \longrightarrow \mathbf{H}^2$ defined by $\kappa(t) := \varphi_x^\lambda(y,t)$ is a F_λ-geodesic. This implies that the reparametrized curve $\alpha : [s_0, s_1] \longrightarrow \mathbf{H}^2$ defined by

$$\alpha(s) := \kappa(t_0 + (s-s_0)(t_1-t_0)/(s_1-s_0))$$

is a F_λ-geodesic too.

Now, since $p_0 = c(s_0)$ and $p_1 = c(s_1)$ are in $\mathcal{D}(r_0)$, we have $t_0, t_1 \in (1/3, 2/3)$ by Step 3, and hence $[t_0, t_1] \subseteq (1/3, 2/3)$. But $\rho \equiv 1$ on $[1/3, 2/3]$ by properties (1) and (3) in Equation 3.4, so we get

$$\kappa(t) = G_x^\lambda(y,t) \quad \text{for all} \quad t \in [t_0, t_1]. \tag{3.8}$$

This leads to considering the reparameterized curve $\overline{c} : [s_0, s_1] \longrightarrow \mathbf{H}^2$ defined by

$$\overline{c}(s) := G_x^\lambda(y, t_0 + (s-s_0)(t_1-t_0)/(s_1-s_0))$$

which is a g_λ-geodesic that satisfies

$$\overline{c}(s_0) = G_x^\lambda(y,t_0) = \kappa(t_0) = p_0 = c(s_0) \quad \text{and} \quad \overline{c}(s_1) = G_x^\lambda(y,t_1) = \kappa(t_1) = p_0 = c(s_1)$$

by Equation 3.8.

Thus, $\overline{c} = c_{|[s_0,s_1]}$ since g_λ has no conjugate points. This writes

$$\begin{aligned} c(s) = \overline{c}(s) &= G_x^\lambda(y, \underbrace{t_0 + (s-s_0)(t_1-t_0)/(s_1-s_0)}_{\in[t_0,t_1]}) \\ &= \kappa(t_0 + (s-s_0)(t_1-t_0)/(s_1-s_0)) \\ &\quad \text{(by Equation 3.8)} \\ &= \alpha(s) \end{aligned}$$

for all $s \in [s_0, s_1]$. Hence $c_{|[s_0,s_1]} = \alpha$, which shows that $c_{|[s_0,s_1]}$ is a F_λ-geodesic (since α is).

As this holds for arbitrary $s_0, s_1 \in I$ with $s_0 < s_1$, we have proved that $c : I \longrightarrow \mathbf{H}^2$ is a F_λ-geodesic.

This establishes Lemma 3.7. $\square$

Proof of Proposition 3.8.
Let $\lambda \in (-\varepsilon, \varepsilon)$ with $\lambda \neq 0$.

As in [2], we use the following theorem of Beltrami to verify that within $\mathcal{D}$ the F_λ-geodesics do not arise as geodesics for a metric diffeomorphic to g_0:

Theorem 3.3 (Beltrami. See [17], Chapter 7, page 26)**.** *If (X, g) is a connected Riemannian manifold such that for every point $p \in X$, there is a chart about p that maps the g-geodesics onto straight lines, then (X, g) has constant sectional curvature.*

Now, if F_λ were Riemannian, then by the boundary rigidity of $(\overline{\mathcal{D}}, g_{0|\overline{\mathcal{D}}})$ given in Theorem 1.3, F_λ would be isometric to F_0 in restriction to $\mathcal{D}$, which would imply that the F_λ-geodesics within $\mathcal{D}$ are diffeomorphically mapped onto straight lines in $\mathbf{R}^2$.

In particular, this would be true for all the F_λ-geodesics within the open set $\mathcal{D}(r_0)$ defined in Lemma 3.7. But this lemma says that every geodesic of the restriction of g_λ to $\mathcal{D}(r_0)$ is a geodesic for F_λ, and therefore the g_λ-geodesics within $\mathcal{D}(r_0)$ would be diffeomorphically mapped onto straight lines in $\mathbf{R}^2$.

Hence the curvature of g_λ would be constant on $\mathcal{D}(r_0)$ by Beltrami's theorem, which is impossible by point (3) in Propositon 3.1. $\square$

References

[1] Álvarez Paiva, J. C., and Berck, G. What is wrong with the Hausdorff measure in Finsler spaces. To appear in *Advances in Mathematics*, 2005.

[2] Arcostanzo, M. Des métriques finslériennes sur le disque à partir d'une fonction distance entre les points du bord. *Comment. Math. Helv. 69*, 2 (1994), 229–248.

[3] Besson, G., Courtois, G., and Gallot, S. Entropies et rigiditiés des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal. 5*, 5 (1995), 731–799.

[4] Besson, G., Courtois, G., and Gallot, S. Minimal entropy and Mostow's rigidity theorems. *Ergodic Theory Dyn. Syst. 16*, 4 (1996), 623–649.

[5] Boland, J., and Newberger, F. Minimal entropy rigidity for Finsler manifolds of negative flag curvature. *Ergodic Theory Dyn. Syst. 21*, 1 (2001), 13–23.

[6] Burago, D., Burago, Y., and Ivanov, S. *A course in metric geometry.* AMS, 2001.

[7] Burago, D., and Ivanov, S. On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume. *Ann. Math. 156*, 3 (2002), 891–914.

[8] Croke, C. Rigidity theorems in Riemannian geometry. *IMA Vol. Math. Appl. 137* (2004), 47–72.

[9] Croke, C., and Sharafutdinov, V. Spectral rigidity of a compact negatively curved manifold. *Topology 37*, 6 (1998), 1265–1273.

[10] Dieudonné, J. *Éléments d'analyse — Tome II (Chapitres XII à XV).* Gauthier-Villars, 1968.

[11] Godbillon, C. *Éléments de topologie algébrique.* Hermann, 1971.

[12] Gromoll, D., Klingenberg, W., and Meyer, W. *Riemannsche Geometrie im Großen.* Springer, 1968.

[13] Guillemin, V., and Pollack, A. *Differential topology.* Prentice-Hall, 1974.

[14] Hirsch, M. *Differential topology.* Springer, 1976.

[15] Ivanov, S. On two-dimensional minimal fillings. *St. Petersbg. Math. J. 13*, 1 (2002), 17–25.

[16] Katok, A. Entropy and closed geodesics. *Ergodic Theory Dyn. Syst. 2*, 3-4 (1982), 339–365.

[17] Spivak, M. *A comprehensive introduction to differential geometry — Volume IV*, second ed. Publish or Perish, 1979.

[18] Verovic, P. Problème de l'entropie minimale pour les métriques de Finsler. *Ergodic Theory Dyn. Syst. 19*, 6 (1999), 1637–1654.

Bruno Colbois, Université de Neuchâtel, Institut de mathématique, Rue Émile Argand 11, Case postale 158, CH–2009 Neuchâtel, Switzerland
E-mail address: bruno.colbois@unine.ch

Florence Newberger, Department of Mathematics, California State University, Long Beach, Long Beach, CA, USA
E-mail address: fnewberg@csulb.edu

Patrick Verovic, UMR 5127 du CNRS & Université de Savoie, Laboratoire de mathématique, Campus scientifique, 73376 Le Bourget-du-Lac Cedex, France
E-mail address: verovic@univ-savoie.fr